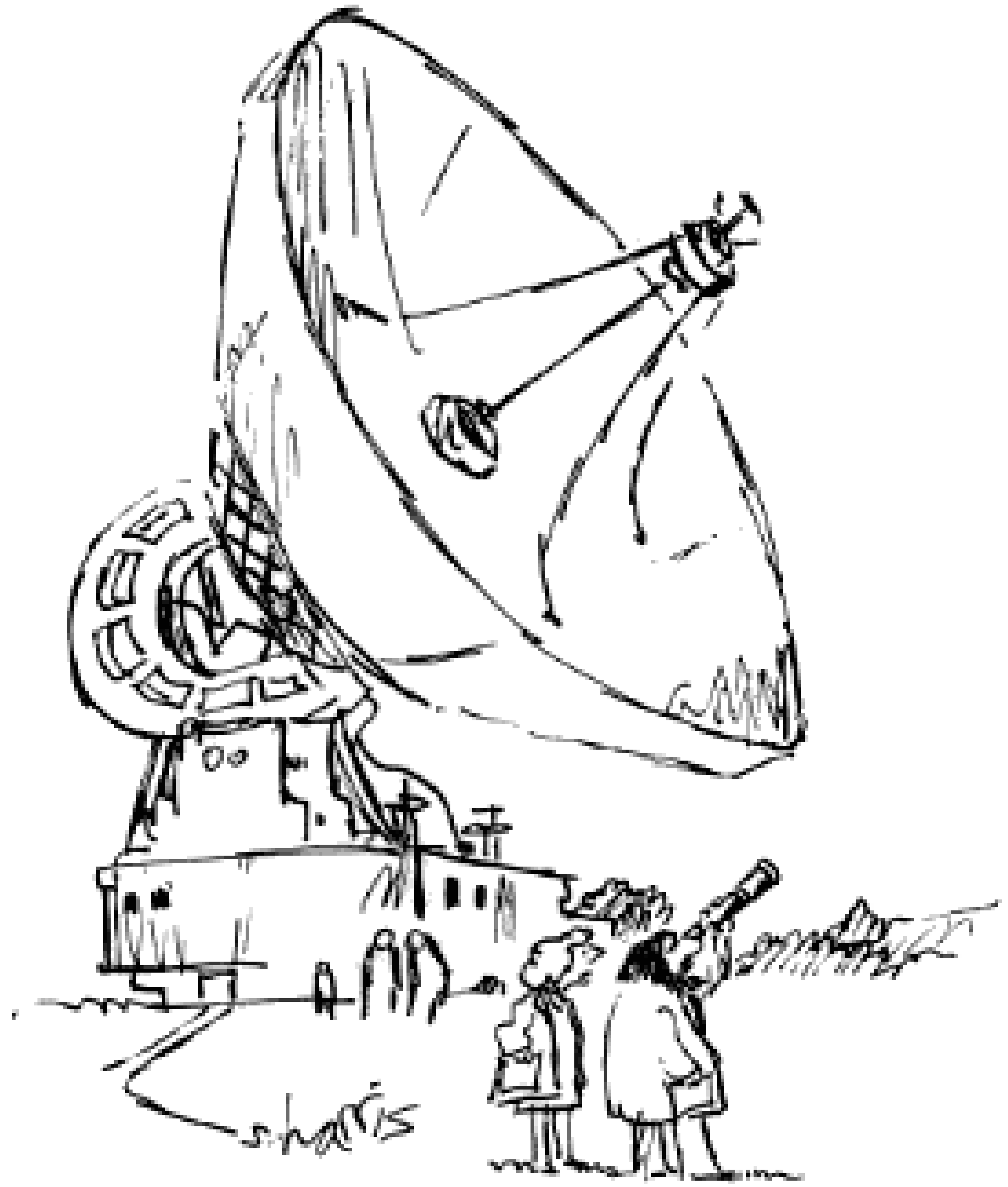
Stereopsis and Epipolar Geometry



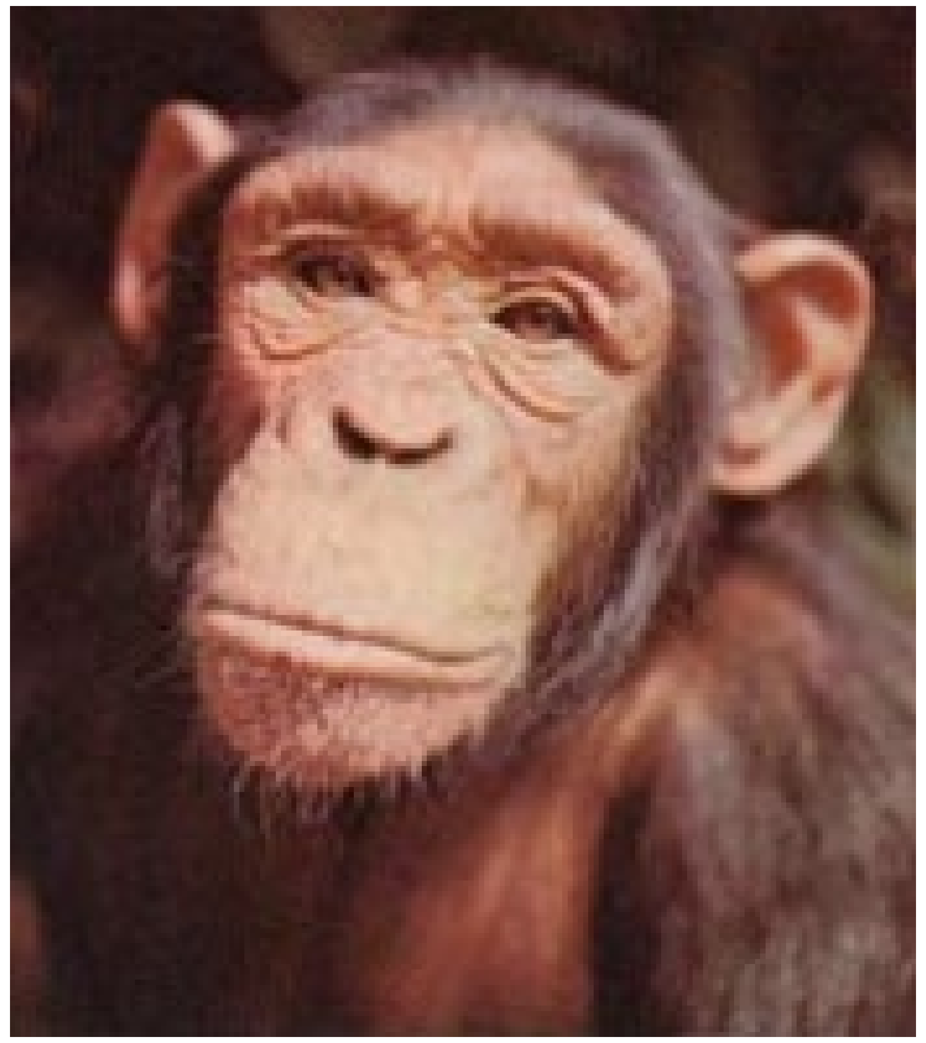
"Just checking."

CS180: : Intro to Comp. Vision and Comp. Photo
Alexei Efros, UC Berkeley, Fall 2024

Vision systems

One camera

Two cameras

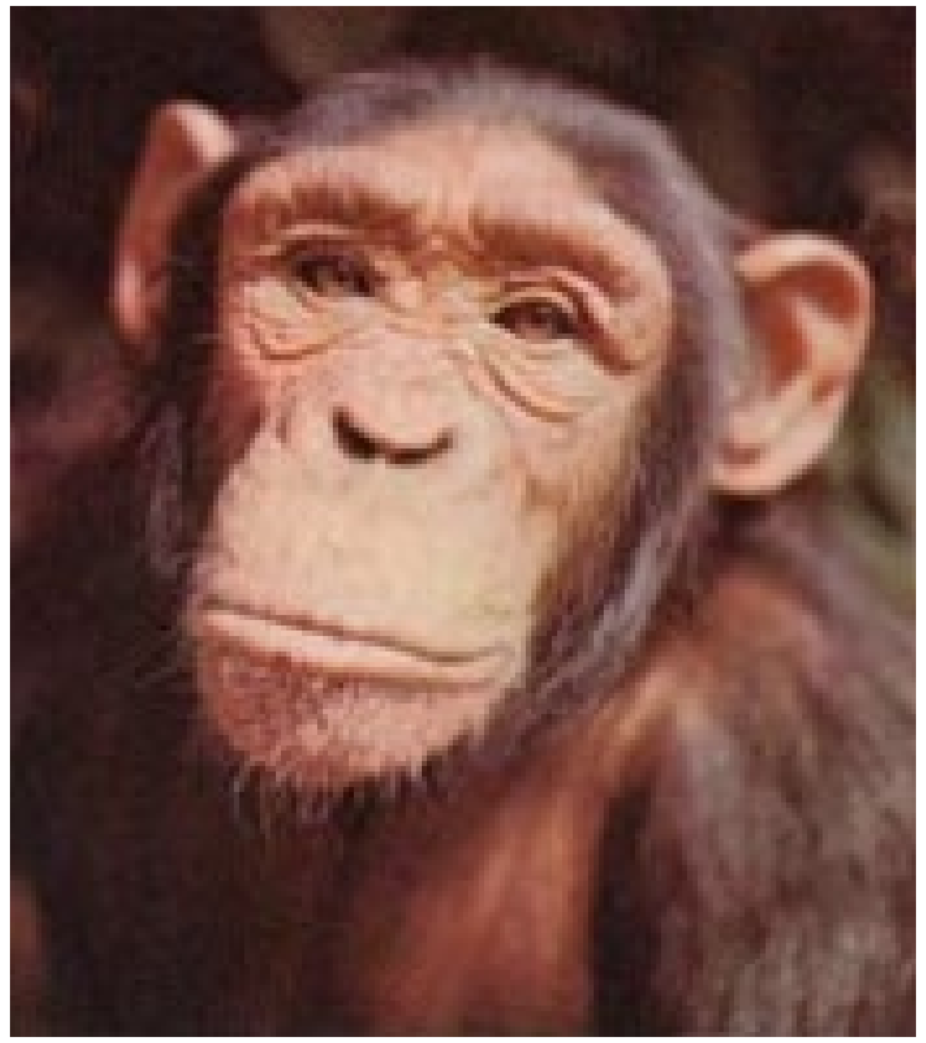


N cameras

Let’s consider two eyes

One camera

Two cameras



N cameras

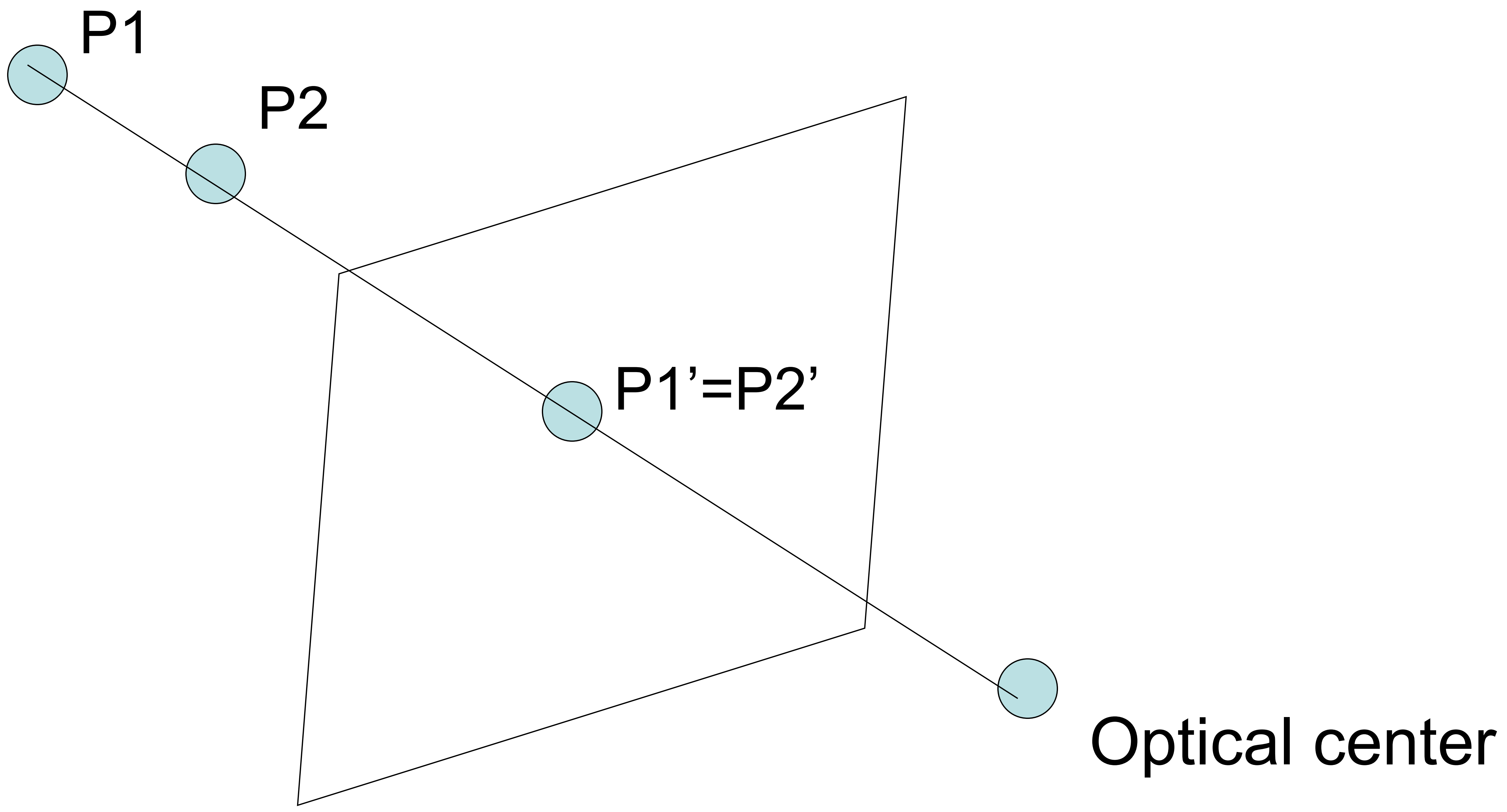
Why multiple views?

- Structure and depth are inherently ambiguous from single views.

Images from Lana Lazebnik

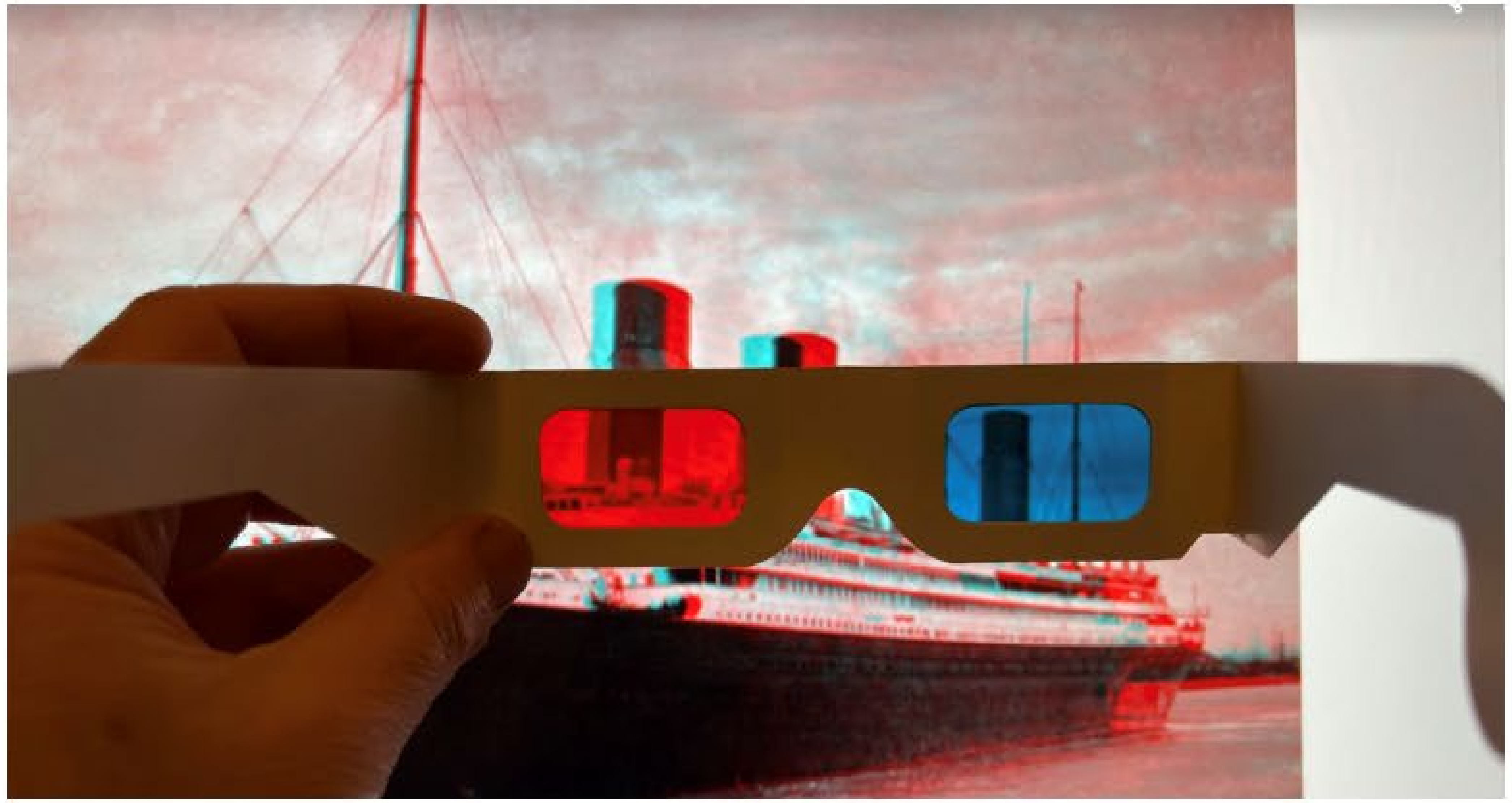
Why multiple views?

- Structure (geometry) and depth are inherently ambiguous from single views.



Stereo images

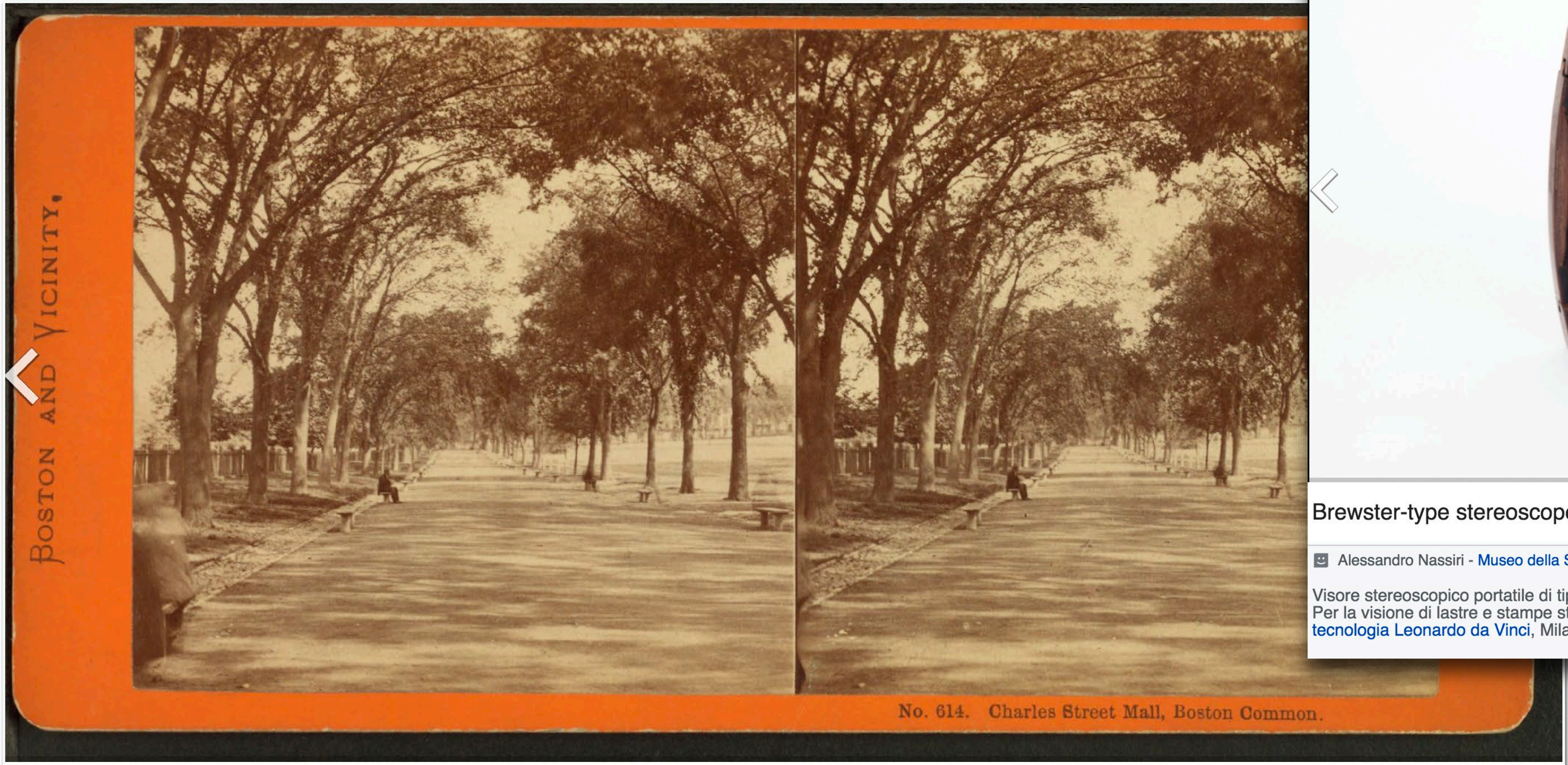
(a)



(b)

Figure 1.1: (a) Stereo anaglyph of the ocean liner, the Titanic [McManus2022]. The red image shows the right eye's view, and cyan the left eye's view. When viewed through stereo red/cyan stereo glasses, as in (b), the cyan contrast appears in the left eye image and the red variations appear to the right eye, creating a the perception of 3d.

Stereoscope



View of Boston*, c. 1860*; an early stereoscopic card for viewing a scene from nature

More details

Soule, John P., 1827-1904 -- Photographer - This image is available from the New York Public Library's Digital Library under the digital ID G90F336_113F: digitalgallery.nypl.org → digitalcollections.nypl.org

File: Charles Street Mall, Boston Common, by Soule, John P., 1827-1904 3.jpg

Created: Coverage: 1860?-1890?. Source Imprint: 1860?-1890?. Digital item published 7-28-2005; updated 4-23-2009.

Brewster-type stereoscope, 1870

More details

Alessandro Nassiri - Museo della Scienza e della Tecnologia "Leonardo da Vinci"

Visore stereoscopico portatile di tipo Brewster, J. Fleury - Hermagis, 1870, con messa a fuoco manuale. Per la visione di lastre e stampe stereoscopiche 8,5x17cm. Museo nazionale della scienza e della tecnologia Leonardo da Vinci, Milano.

File: IGB 006055 Visore stereoscopico portatile Museo scienza e tecnologia Milano.jpg

Created: 1 July 2014

Depth without objects

Random dot stereograms (Bela Julesz)

1	0	1	0	1	0	0	1	0	1
1	0	0	1	0	1	0	1	0	0
0	0	1	1	0	1	1	0	1	0
0	1	0	Y	A	A	B	B	0	1
1	1	1	X	B	A	B	A	0	1
0	0	1	X	A	A	B	A	1	0
1	1	1	Y	B	B	A	B	0	1
1	0	0	1	1	0	1	1	0	1
1	1	0	0	1	1	0	1	1	1
0	1	0	0	0	1	1	1	1	0

1	0	1	0	1	0	0	1	0	1
1	0	0	1	0	1	0	1	0	0
0	0	1	1	0	1	1	0	1	0
0	1	0	A	A	B	B	X	0	1
1	1	1	B	A	B	A	Y	0	1
0	0	1	A	A	B	A	Y	1	0
1	1	1	B	B	A	B	X	0	1
1	0	0	1	1	0	1	1	0	1
1	1	0	0	1	1	0	1	1	1
0	1	0	0	0	1	1	1	1	0

Julesz, 1971

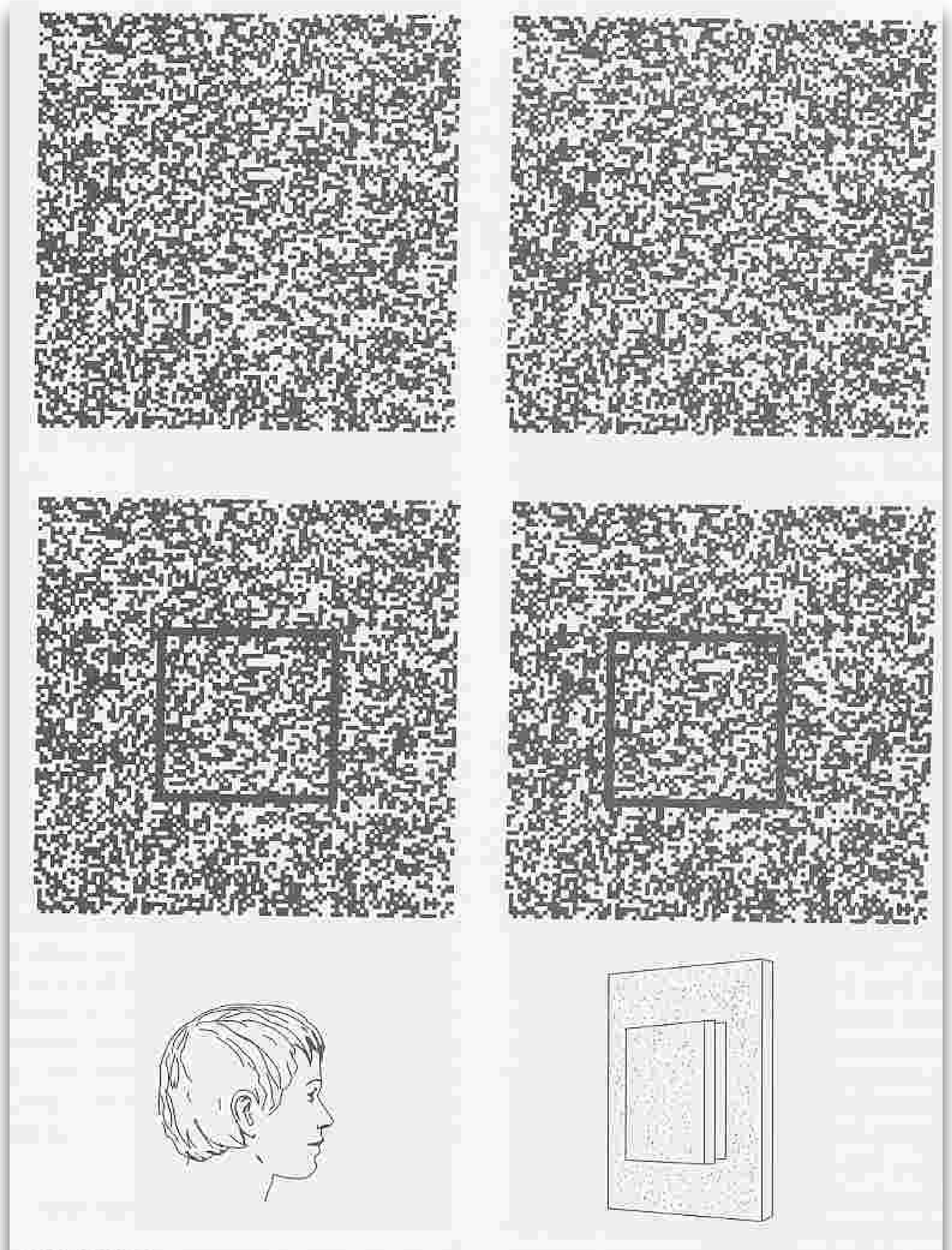
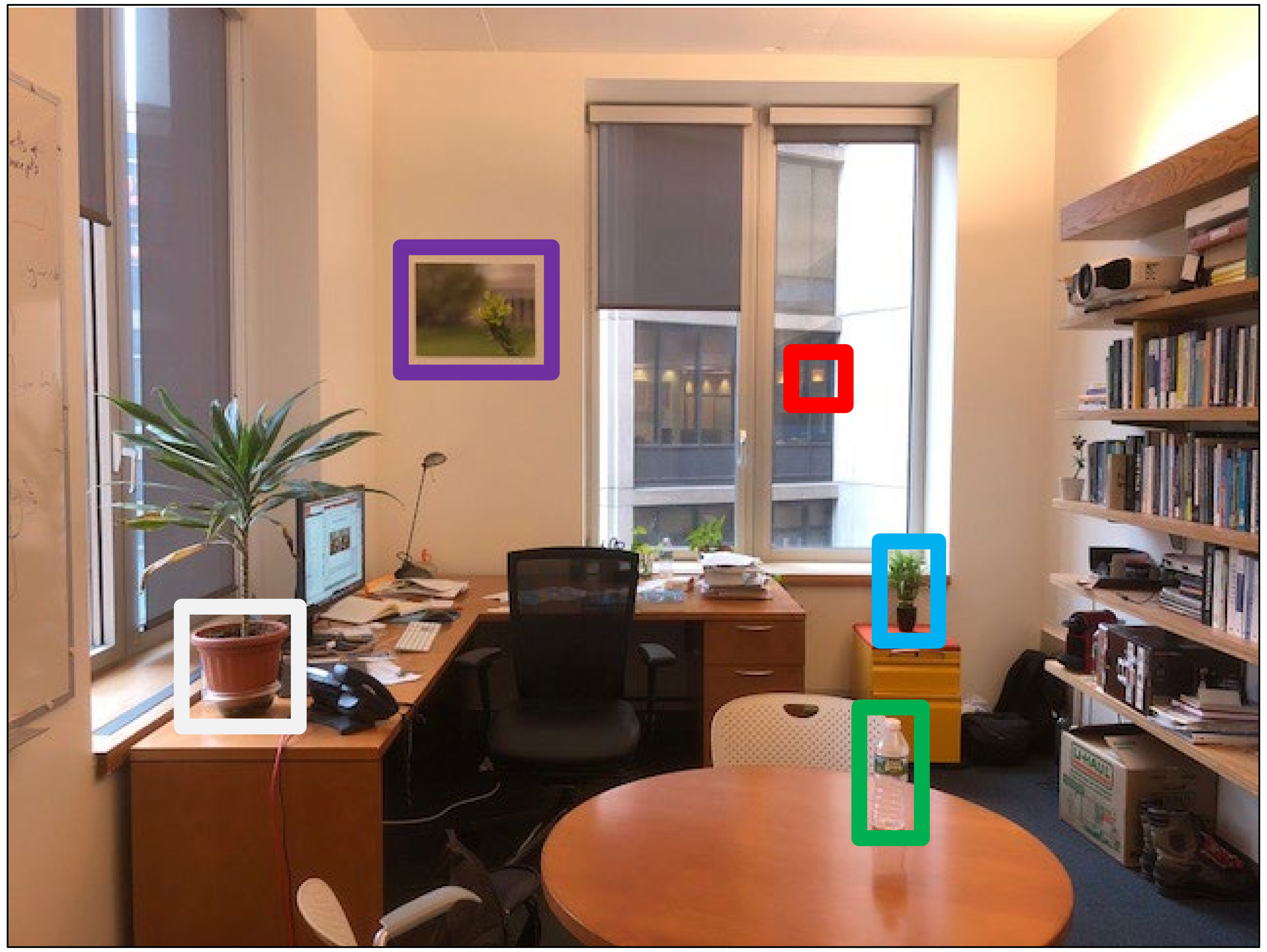


FIGURE 8.13

disparity

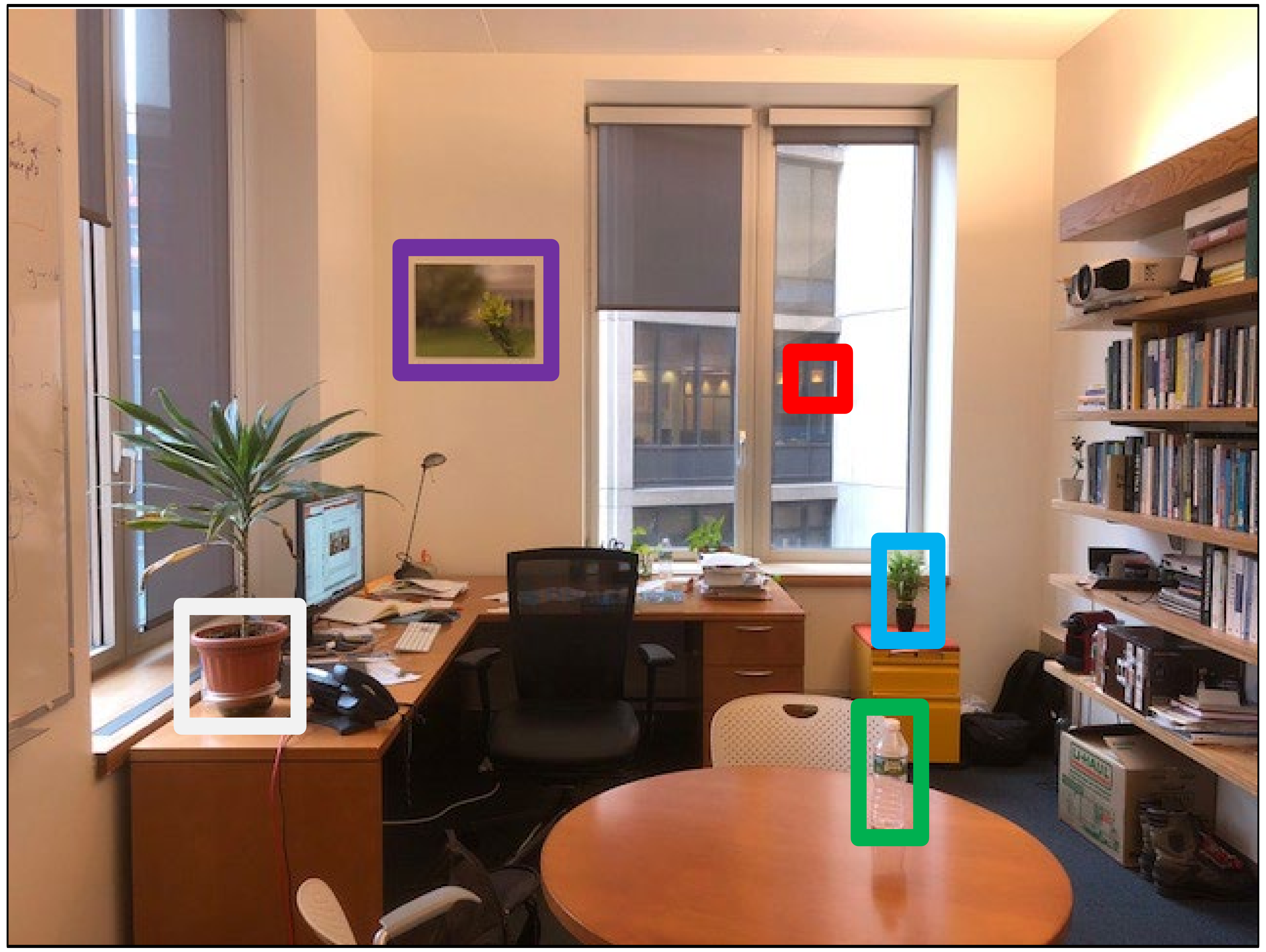


Left image

Right image

Antonio took one picture, then he moved ~1m to the right and took a second picture.

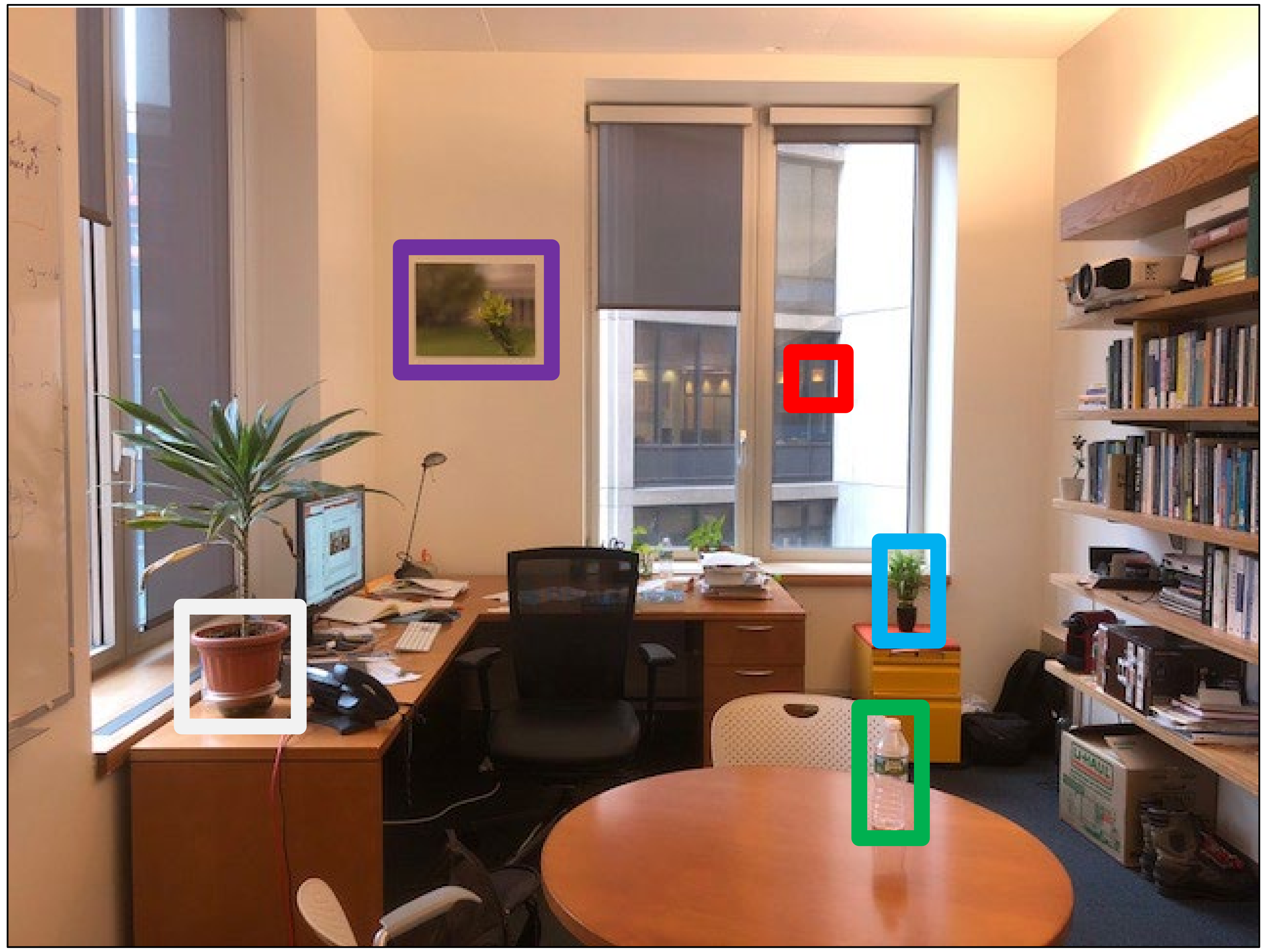
disparity



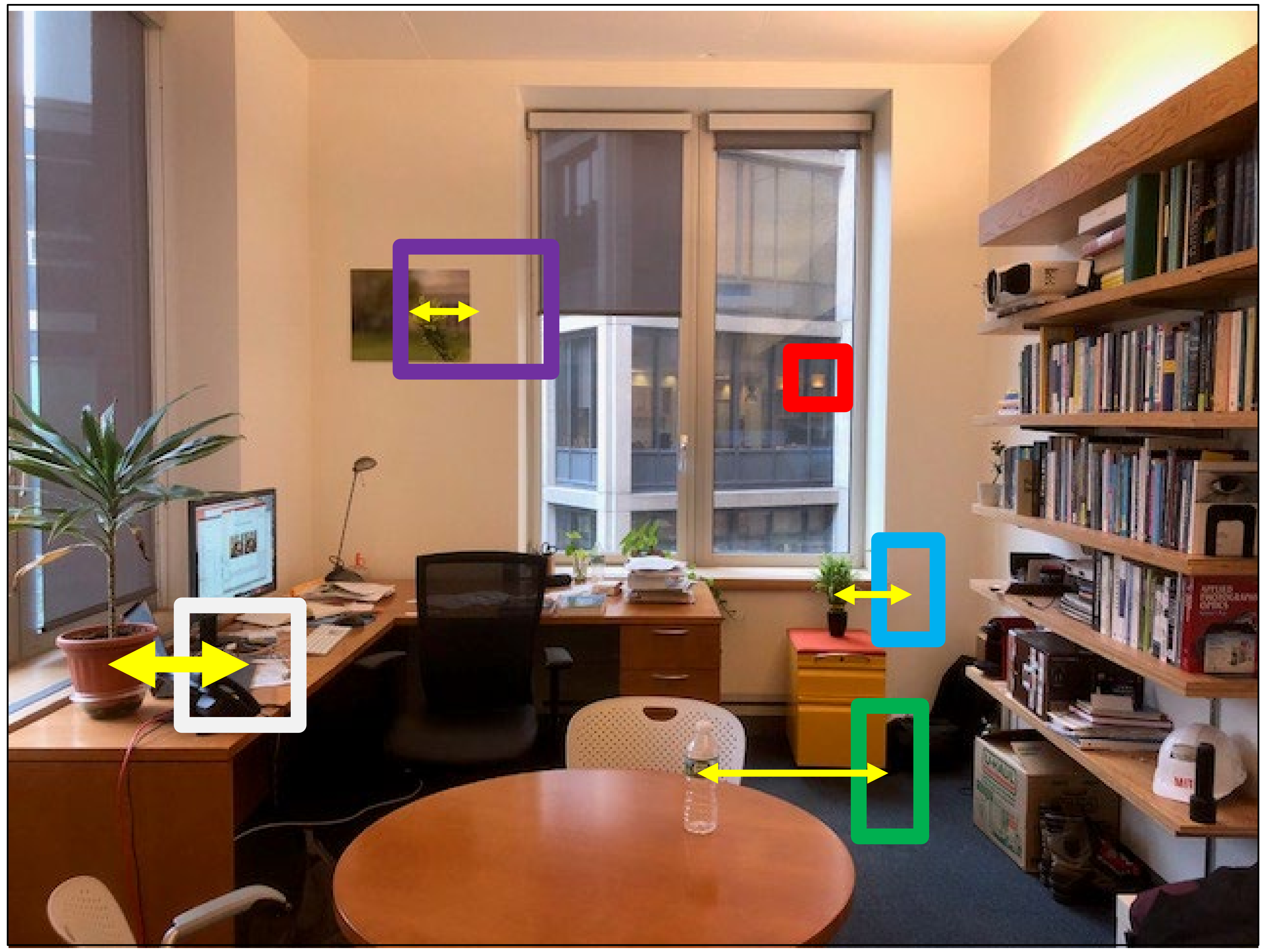
Left image

Right image

disparity



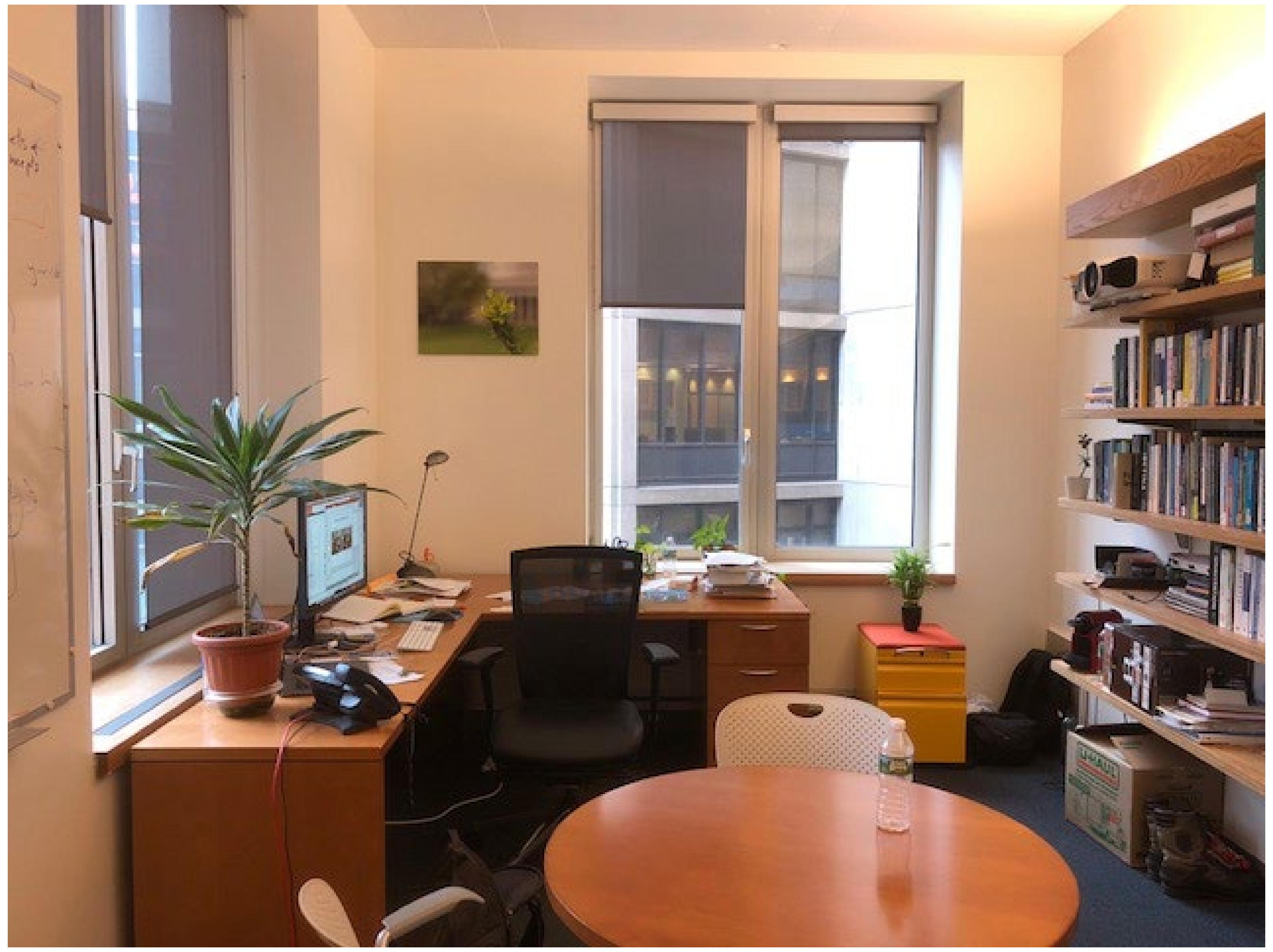
Left image



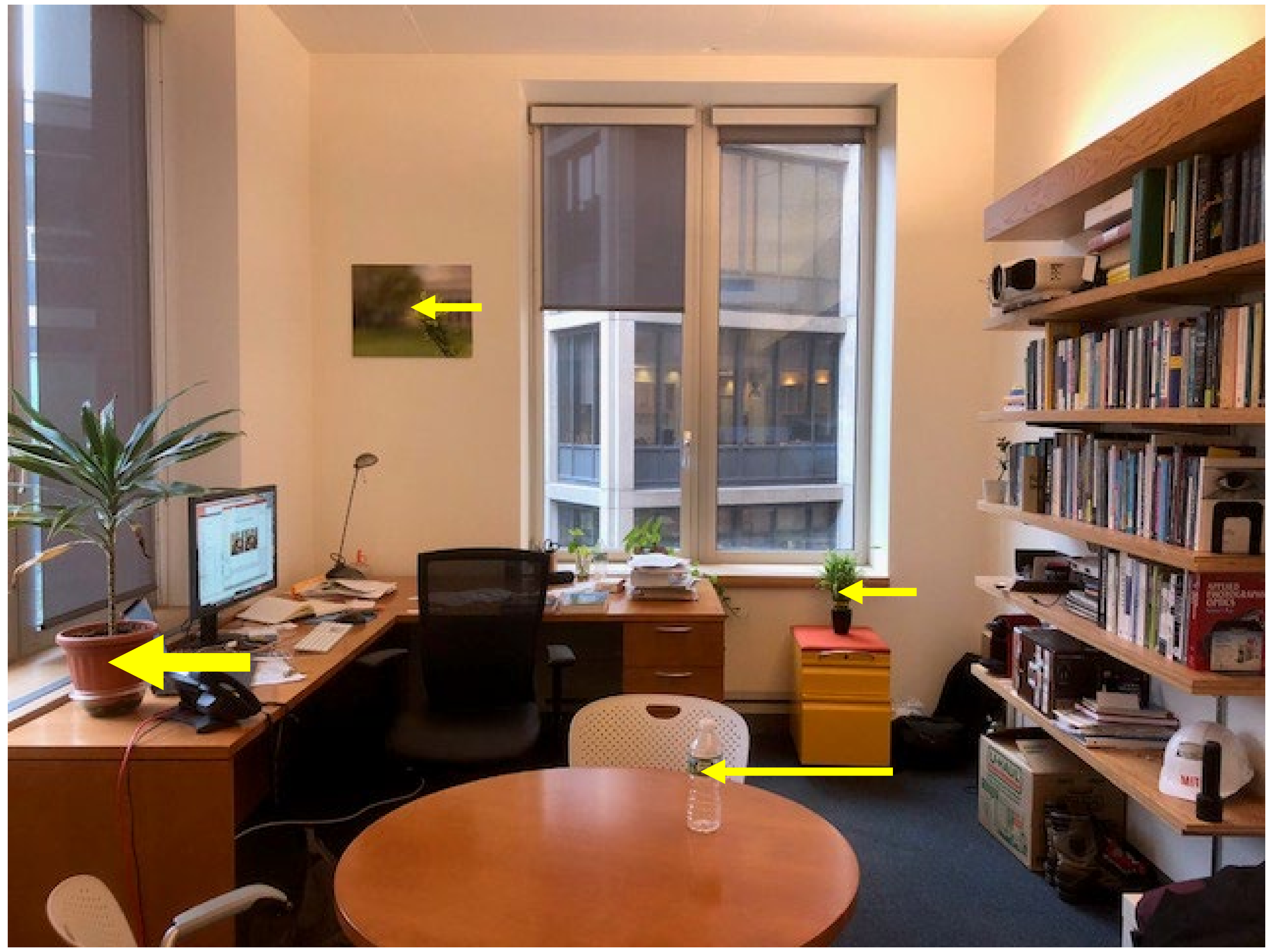
Right image

Disparity map

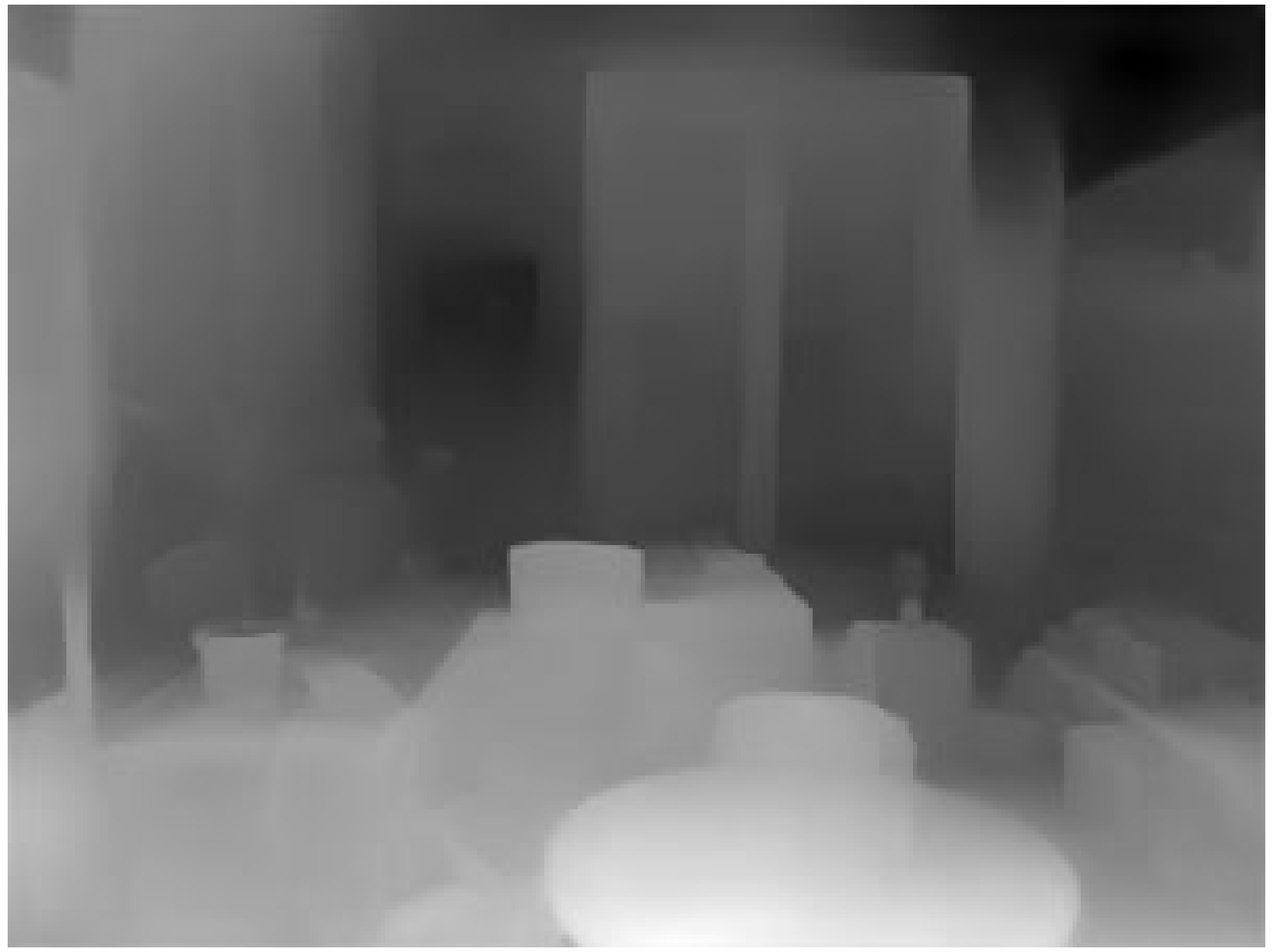
I(x,y)



$I'(x,y) = I(x+D(x,y), y)$

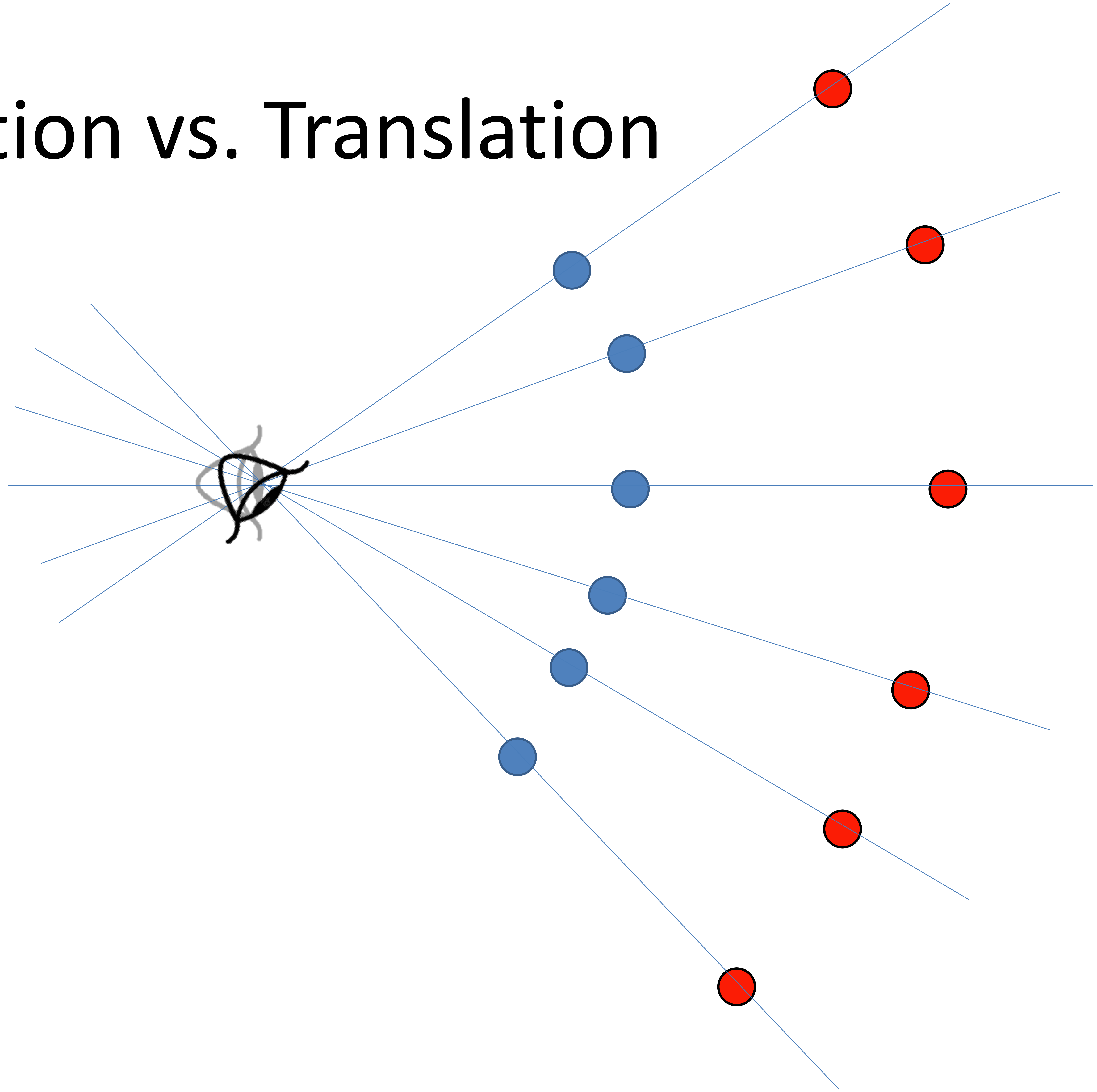


D(x,y)

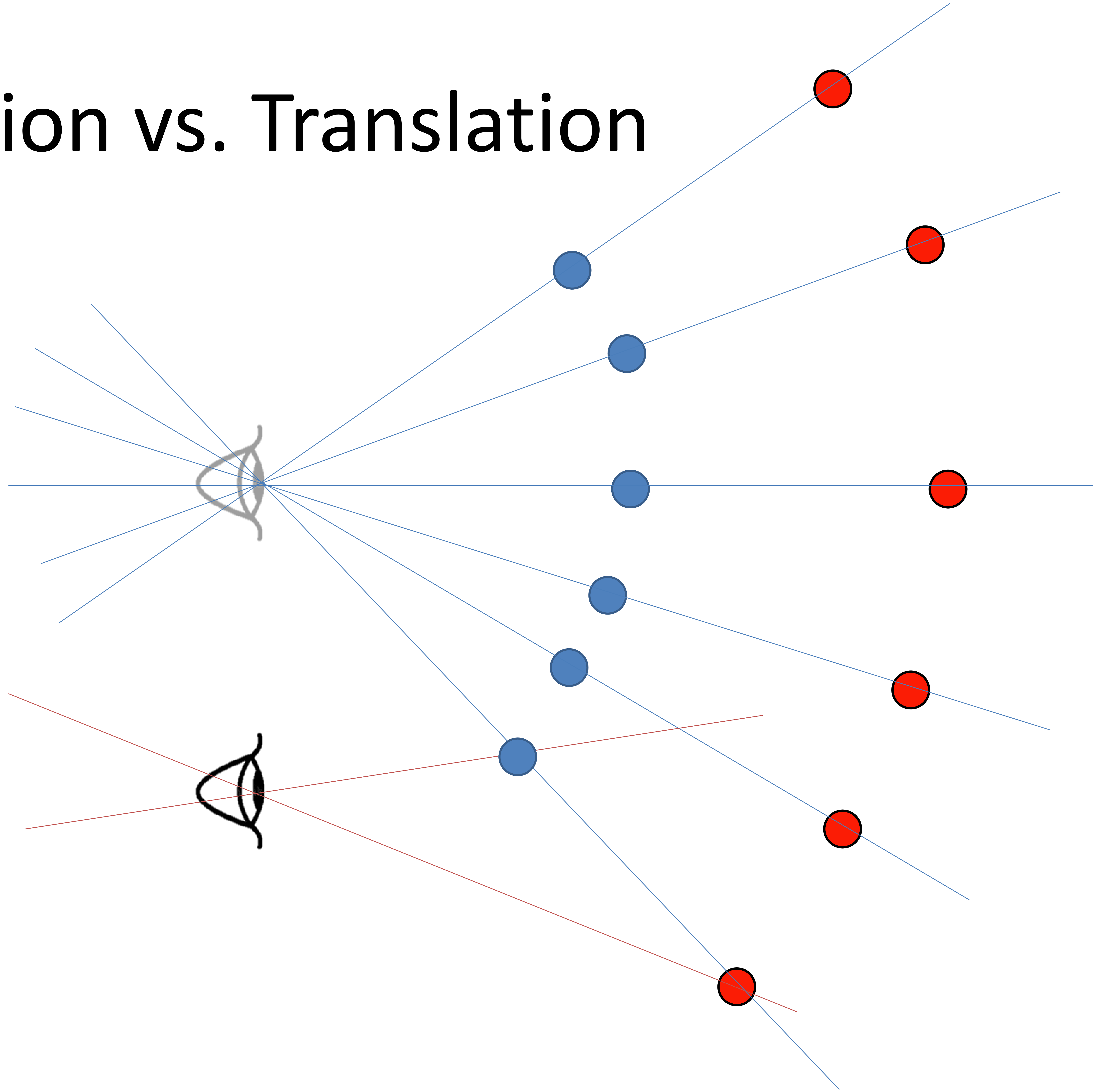


$$Z(x,y) = \frac{a}{D(x,y)}$$

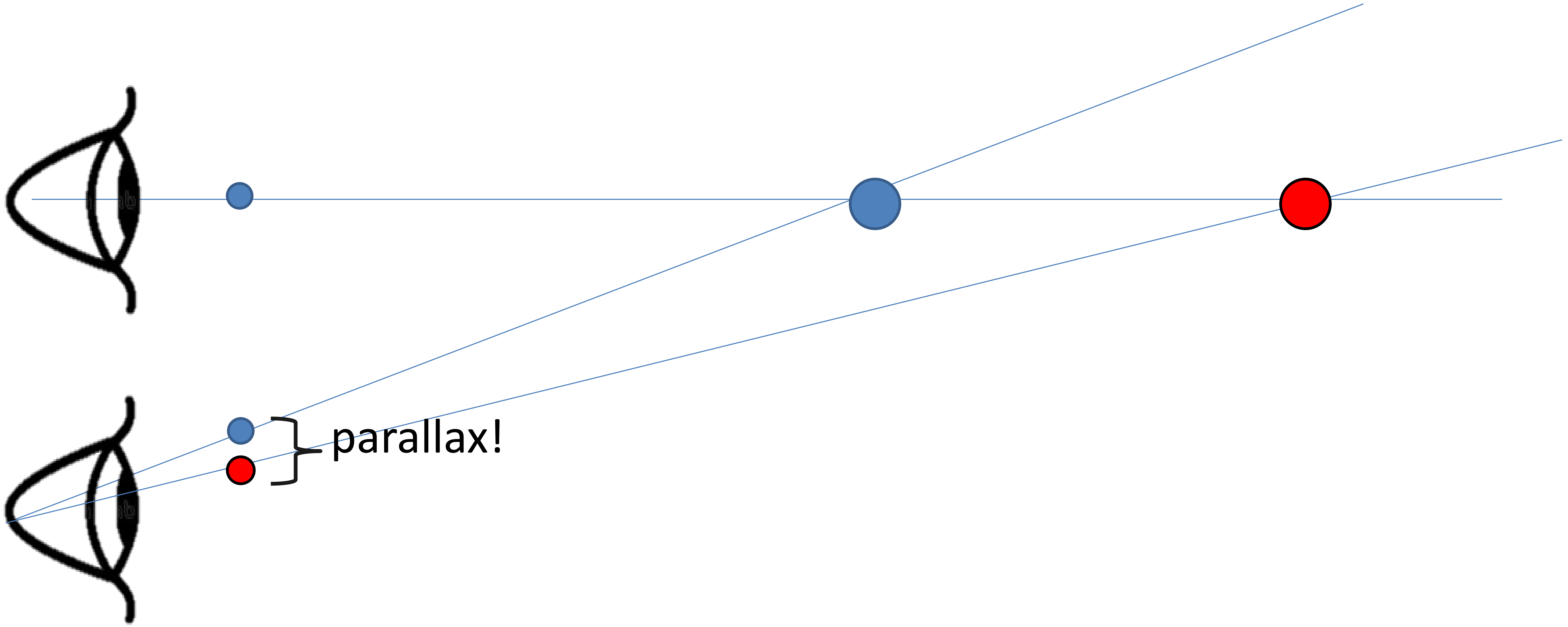
Rotation vs. Translation



Rotation vs. Translation



Parallax



Parallax = *from ancient Greek parállaxis*
= *Para* (side by side) + *allássō*, (to alter)
= *Change in position from different view point*

Two eyes give you parallax, you can also move to see more parallax = “Motion Parallax”

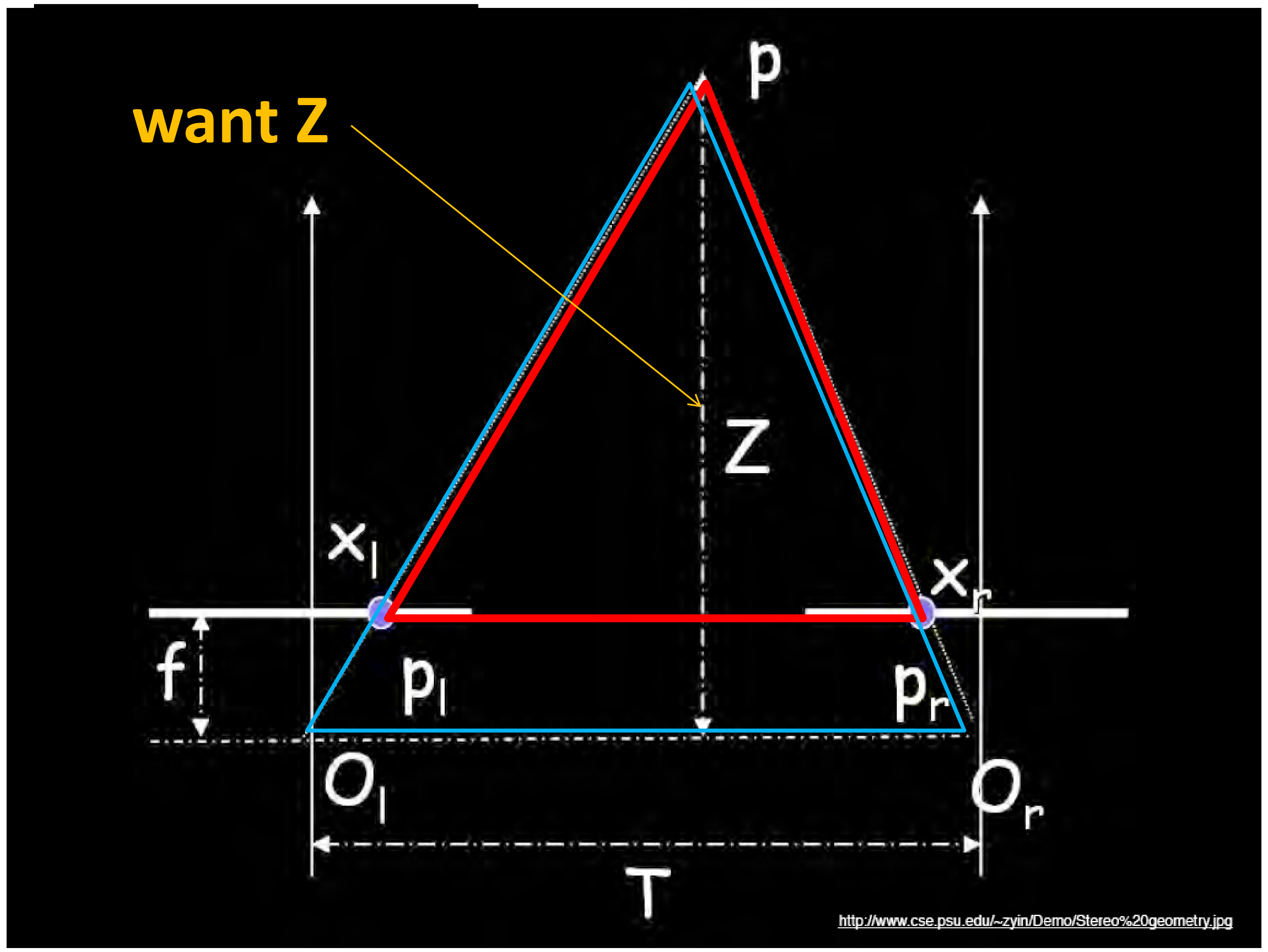
Stereo vision

Two cameras, simultaneous views

Single moving camera and static scene

Geometry for a simple stereo system

- Assume **parallel** optical axes, known camera parameters (i.e., calibrated cameras).



Use similar triangles (p_l, P, p_r) and (O_l, P, O_r):

$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$

$$Z = f \frac{T}{x_r - x_l}$$

disparity

Grauman

Non-parametric transformation!

image I(x,y)

Disparity map D(x,y)

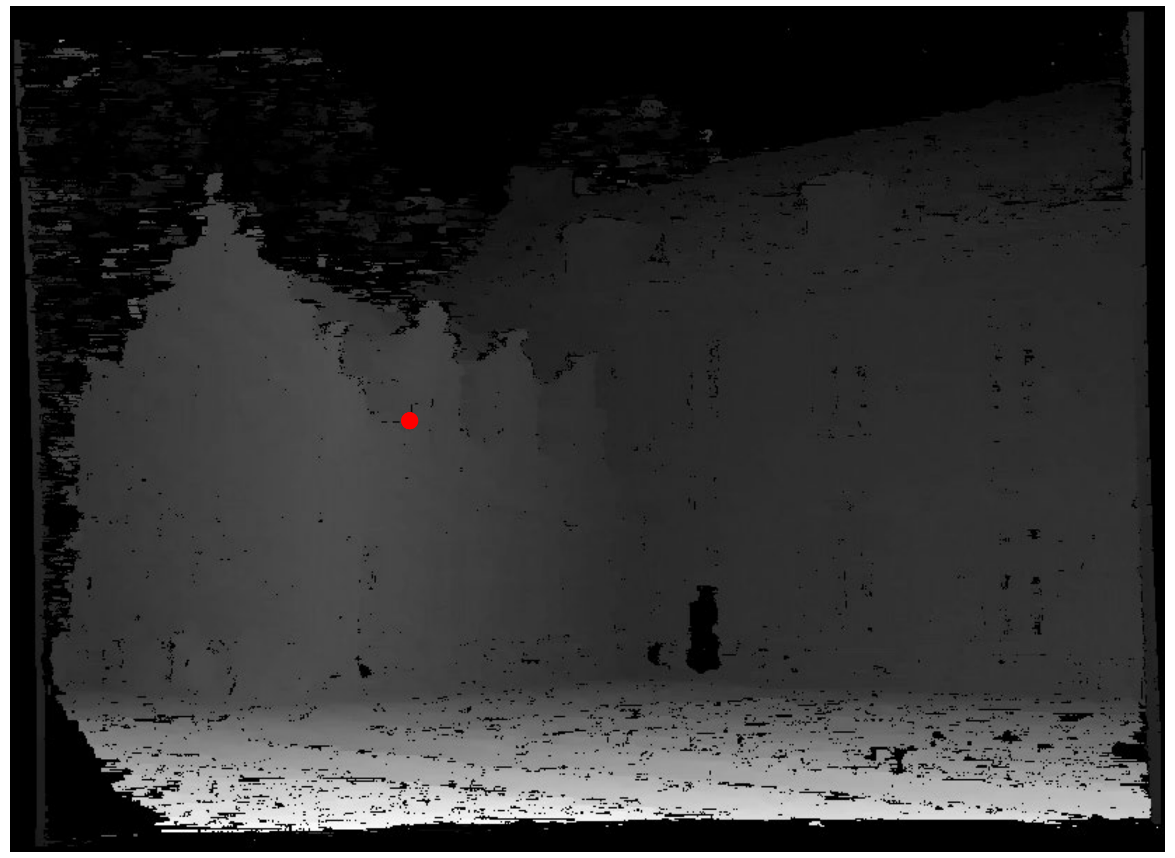


image I´(x´,y´)

$$(x',y')=(x+D(x,y),\ y)$$

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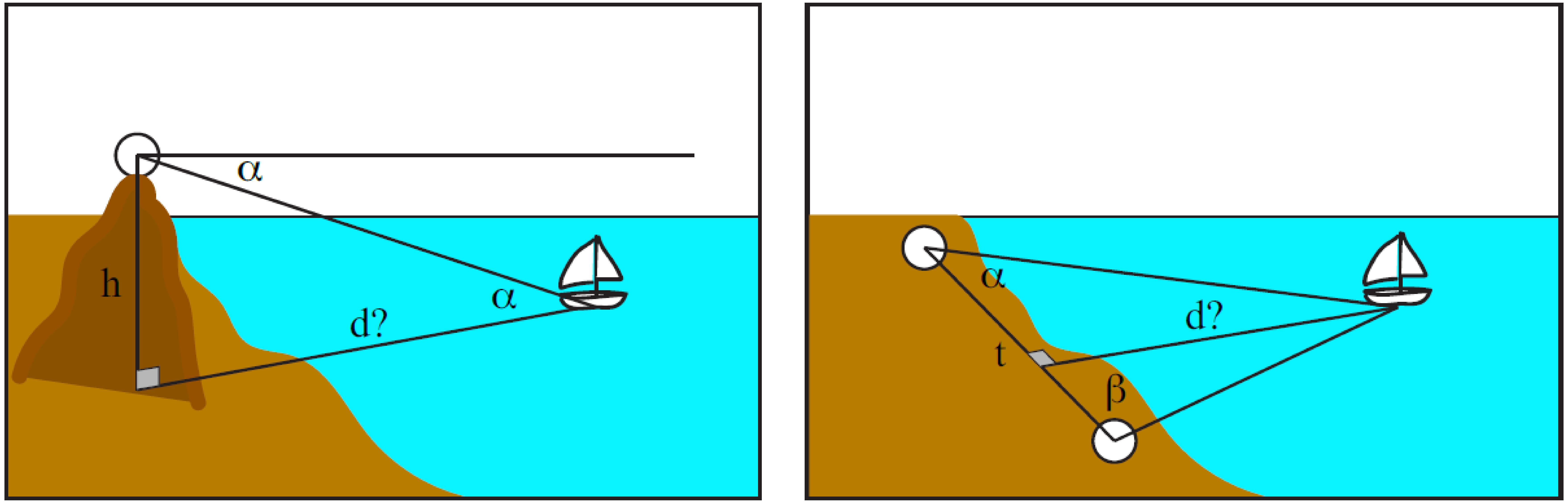
Triangulation for Ship Navigation

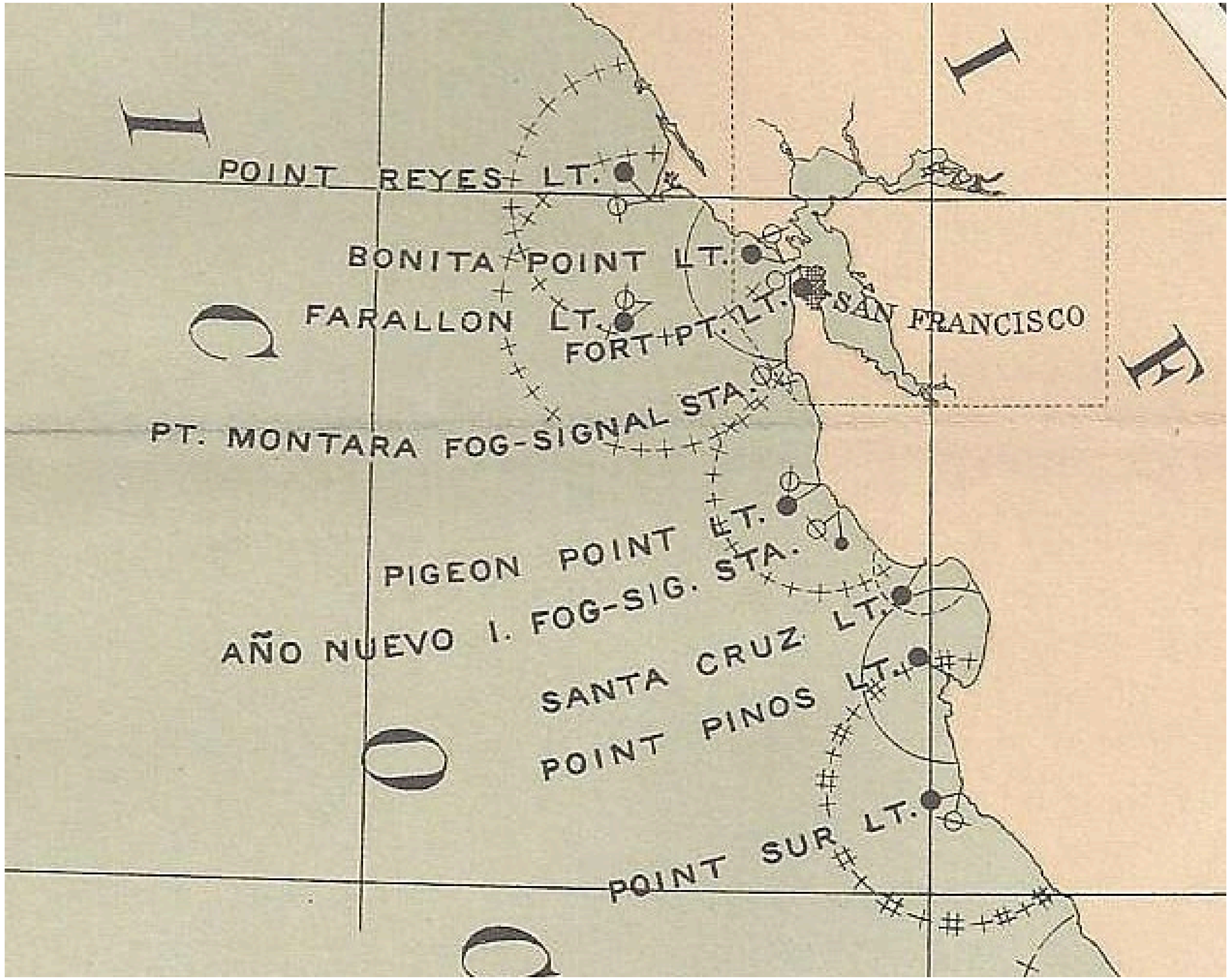
Figure 40.2: Two methods to estimate the distance of a boat from the coast. (left) The first method uses a single observation point, with knowledge of the observer's height above the water. (right) The second method uses two observation points.

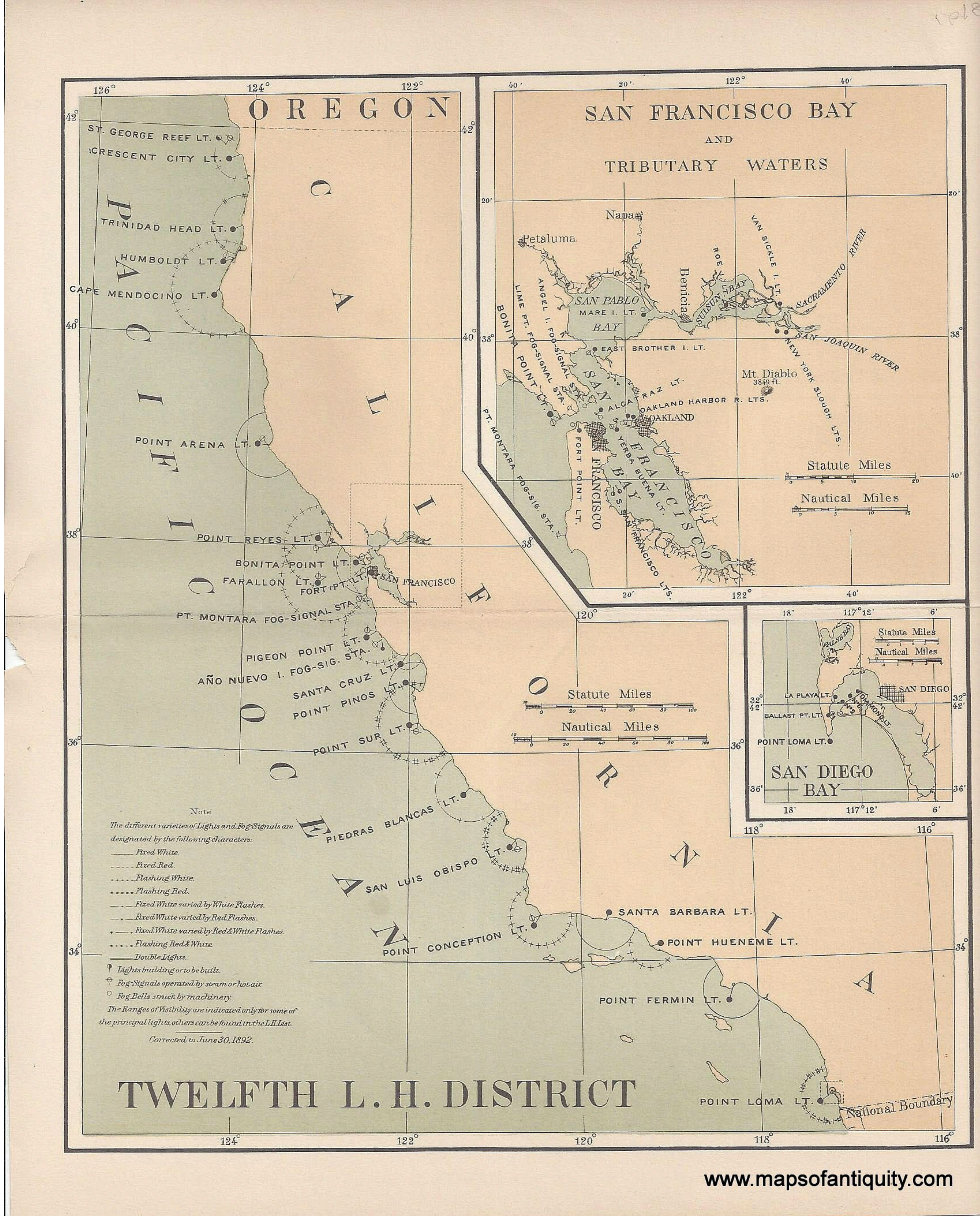


$$d = \frac{h}{\tan(\alpha)}$$

$$d = t\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha+\beta)}$$

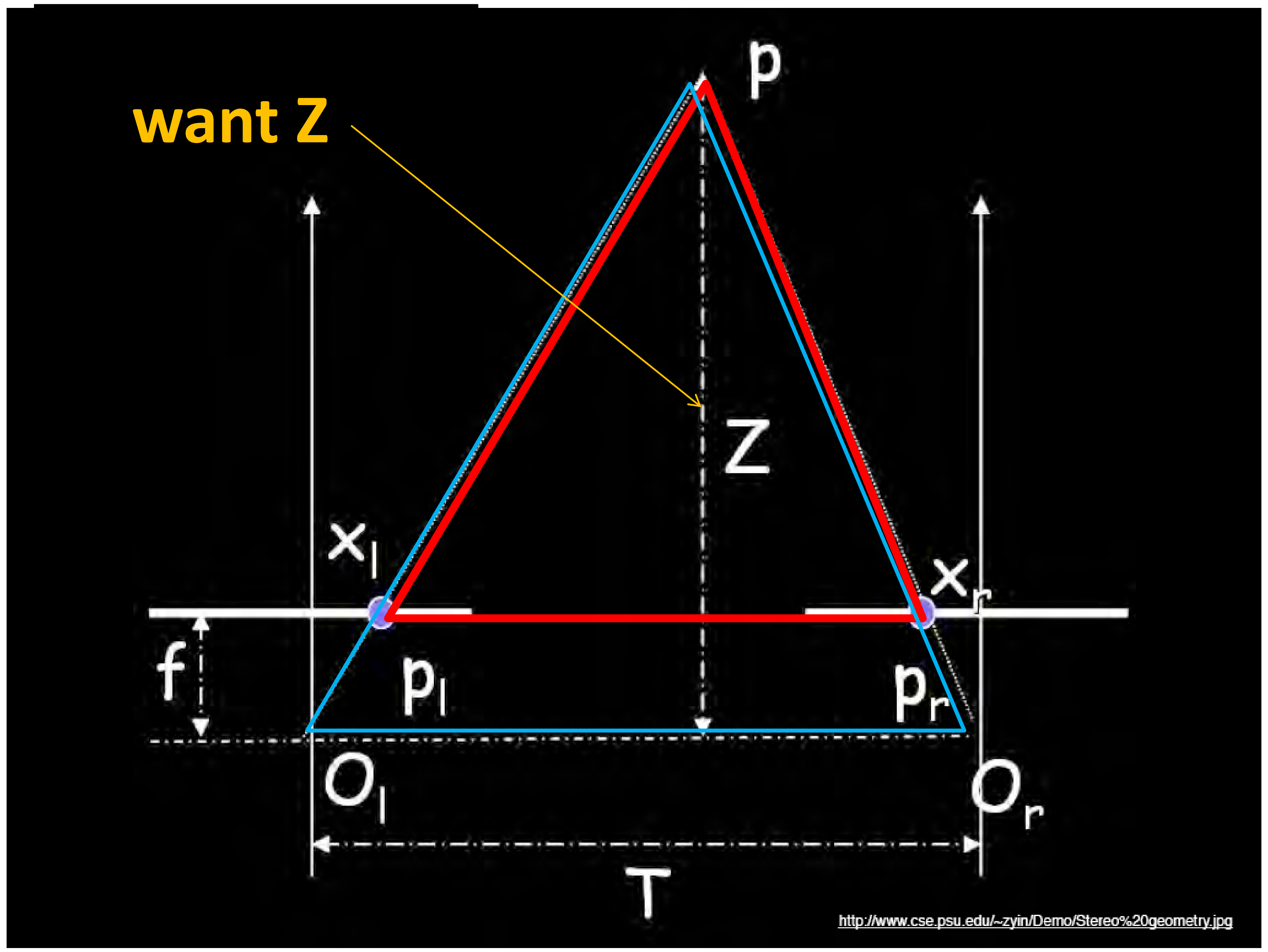
Triangulation





Geometry for a simple stereo system

- Assume **parallel** optical axes, known camera parameters (i.e., calibrated cameras).



Use similar triangles (p_l, P, p_r) and (O_l, P, O_r):

$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$

$$Z = f \frac{T}{x_r - x_l}$$

disparity

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Correspondence problem

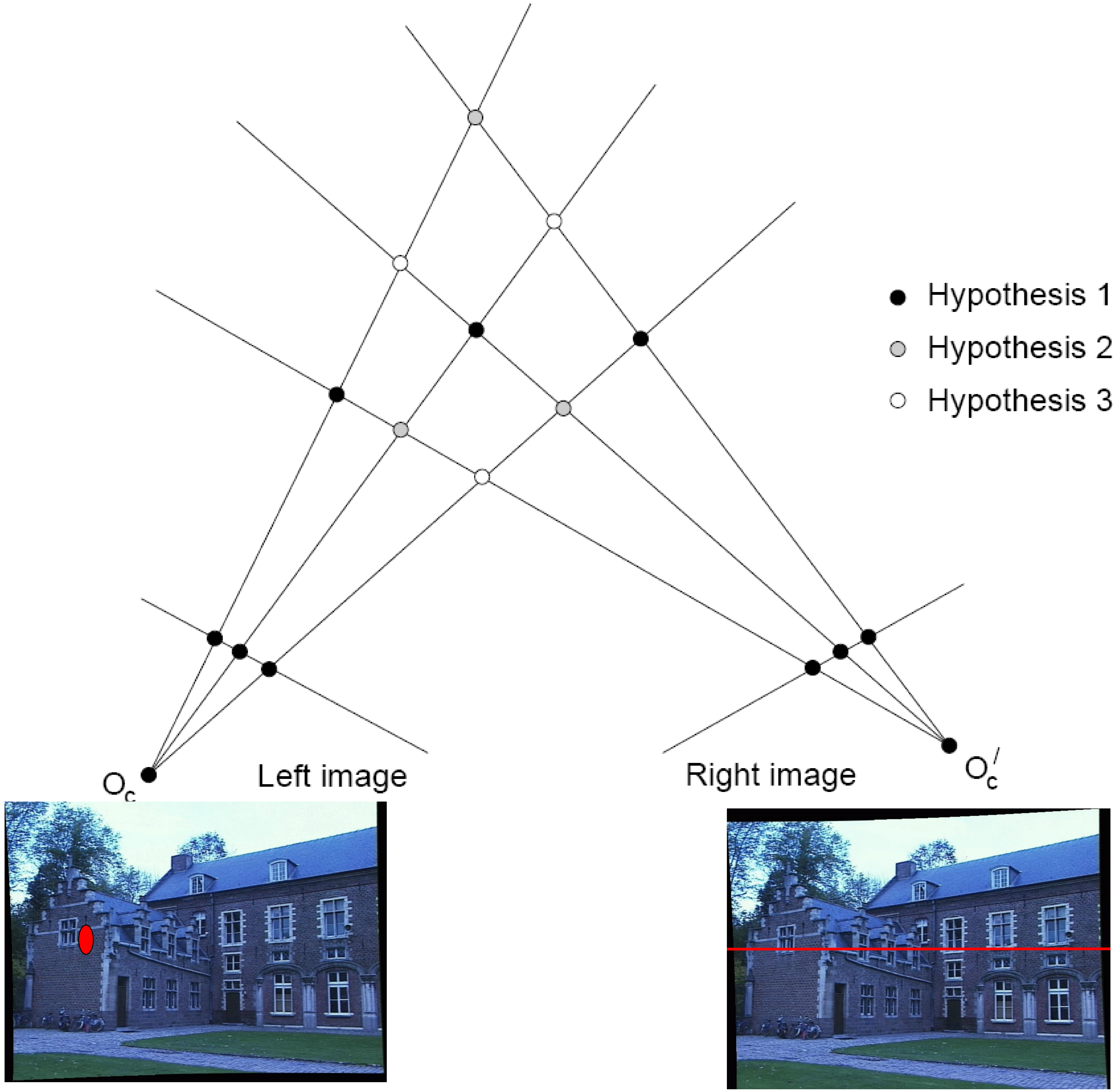

Figure from Gee & Cipolla 1999

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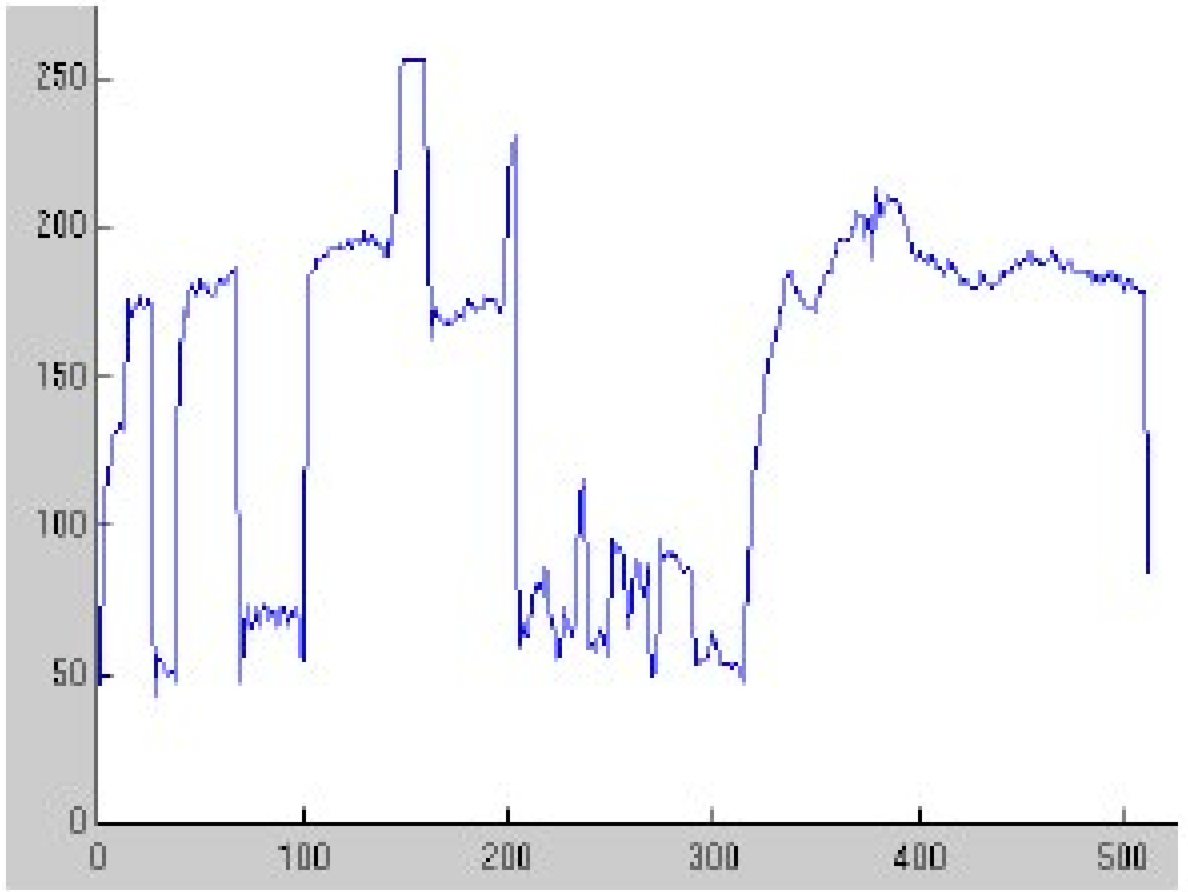
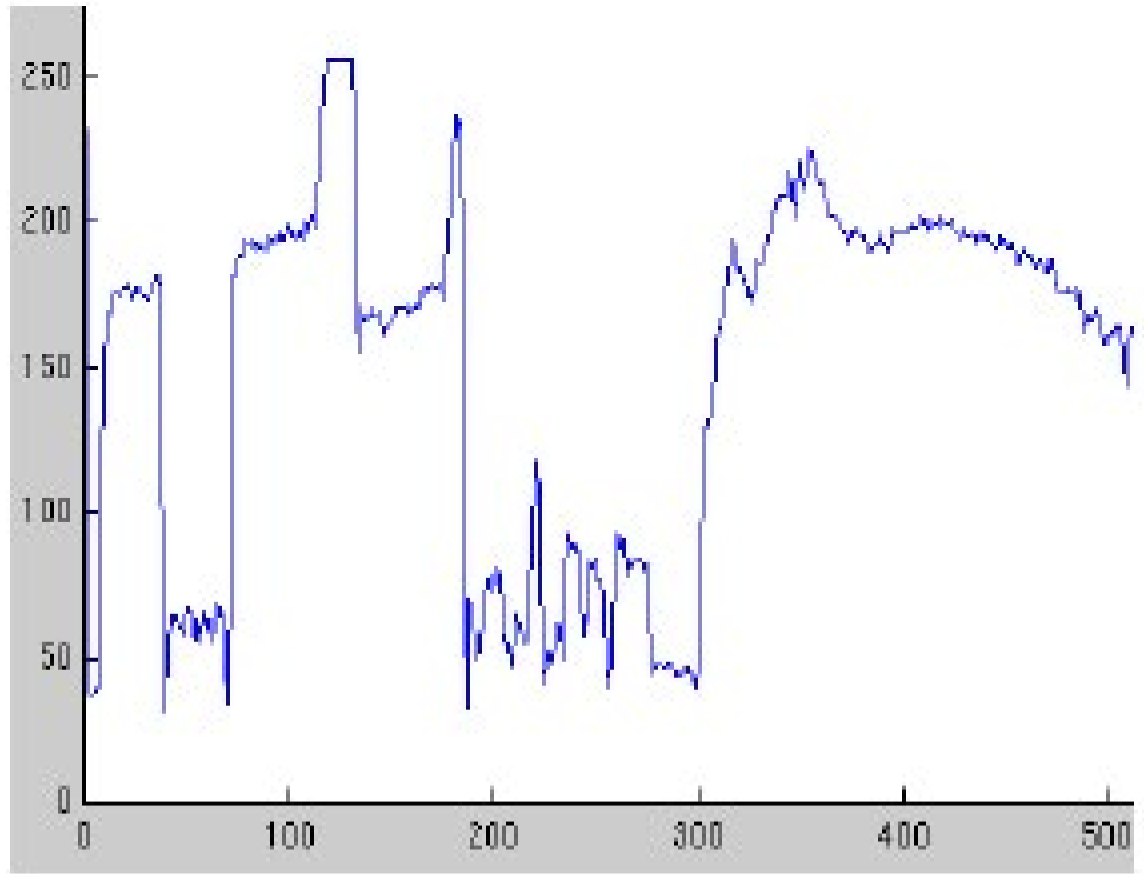
Intensity profiles

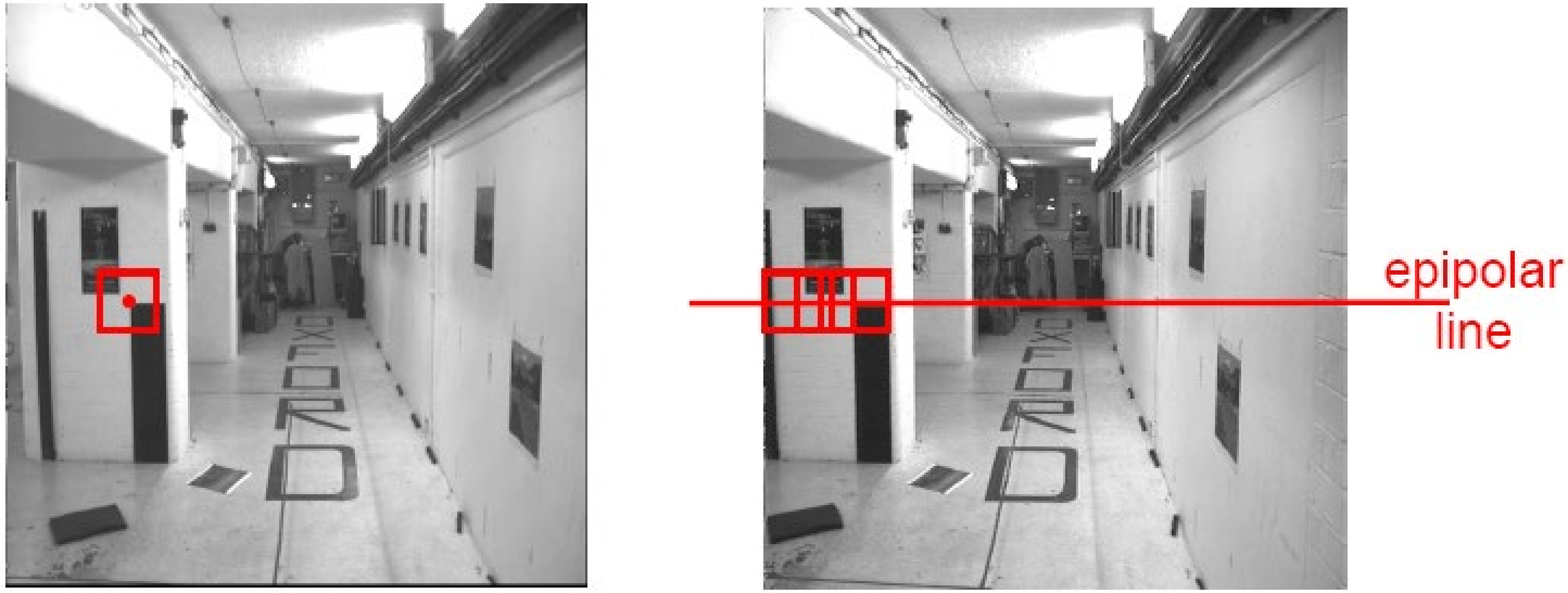




- Clear correspondence between intensities, but also noise and ambiguity

Source: Andrew Zisserman

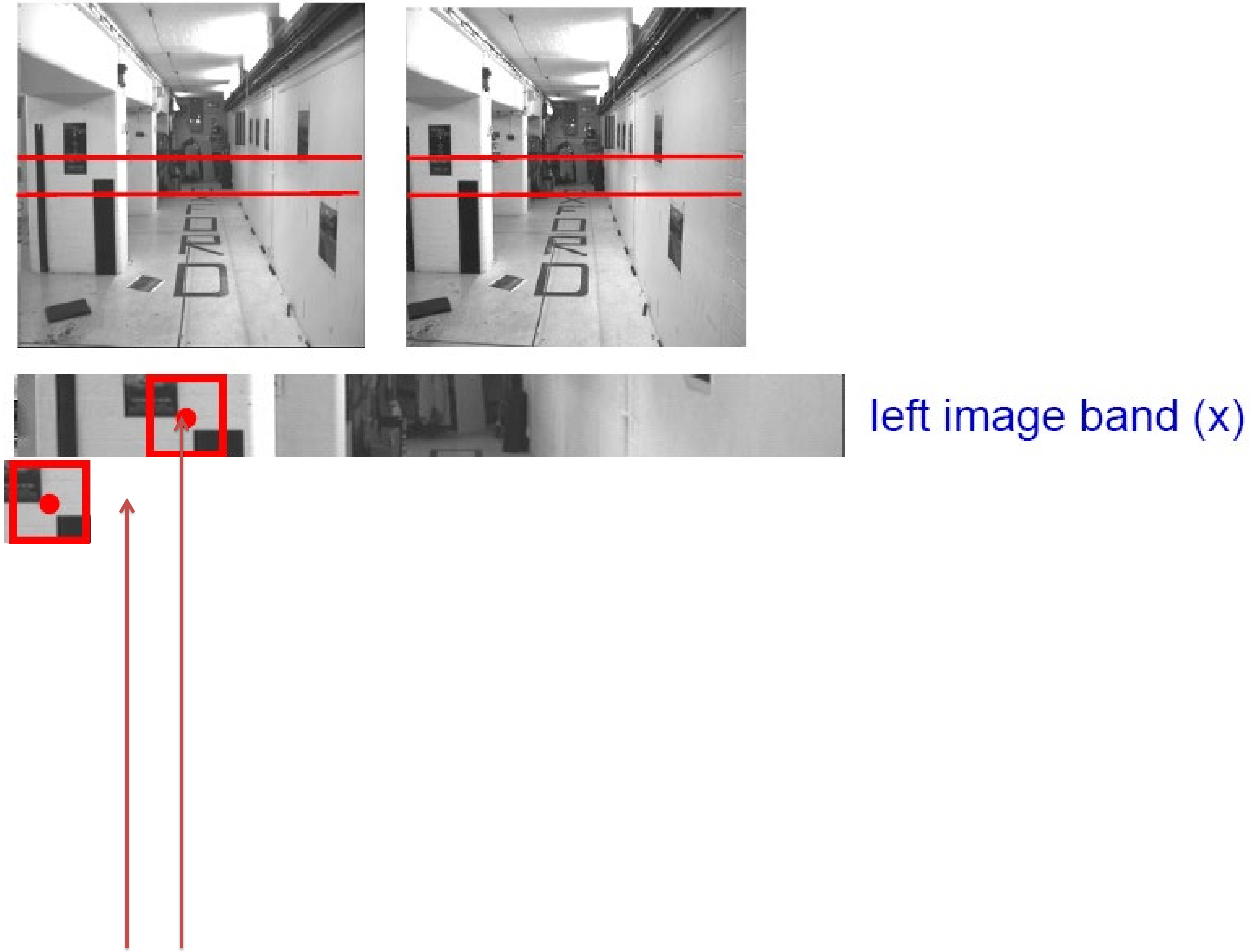
Correspondence problem



Neighborhood of corresponding points are similar in intensity patterns.

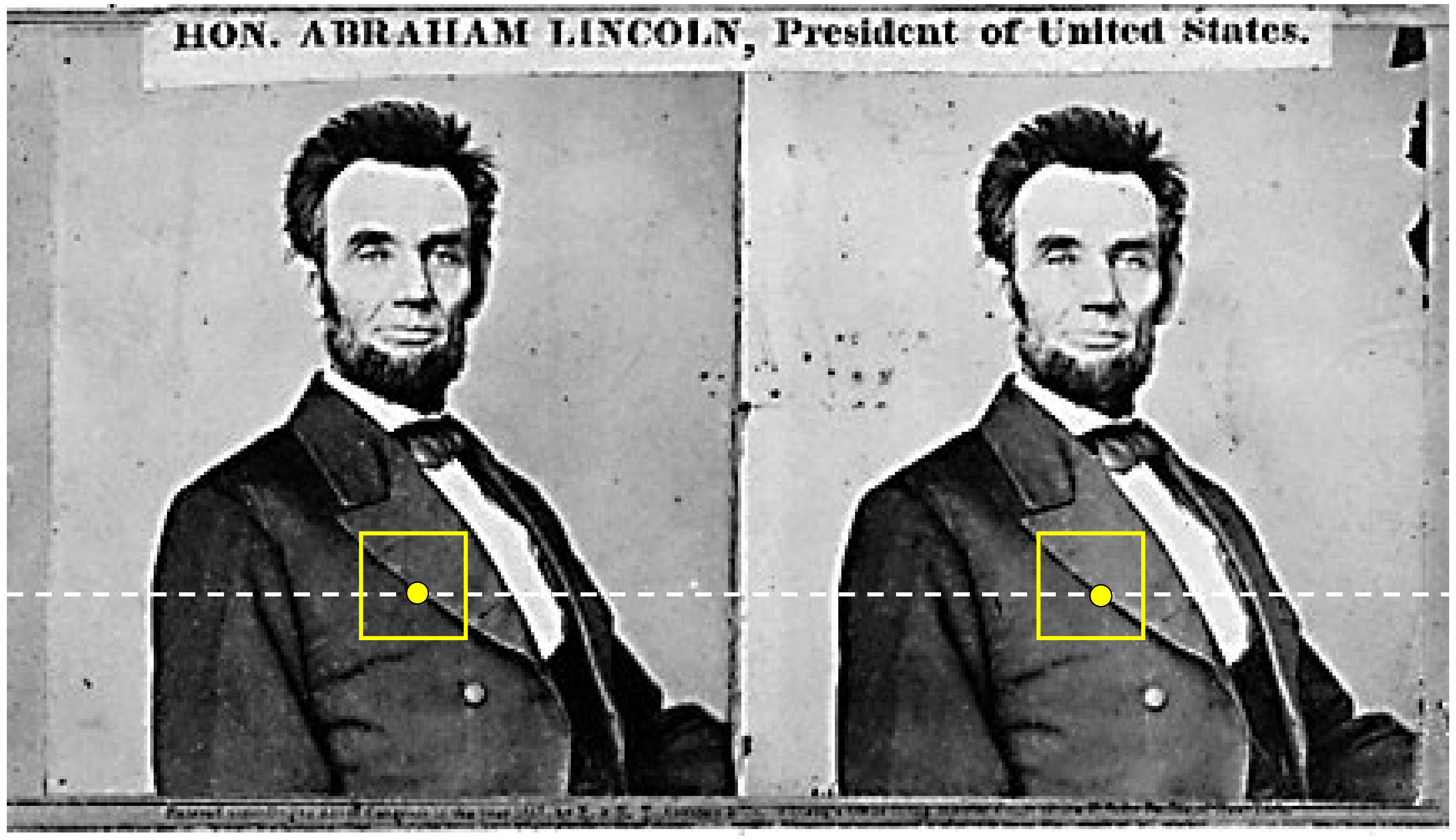
Source: Andrew Zisserman

Correlation-based window matching



Source: Andrew Zisserman

Dense correspondence search



For each epipolar line

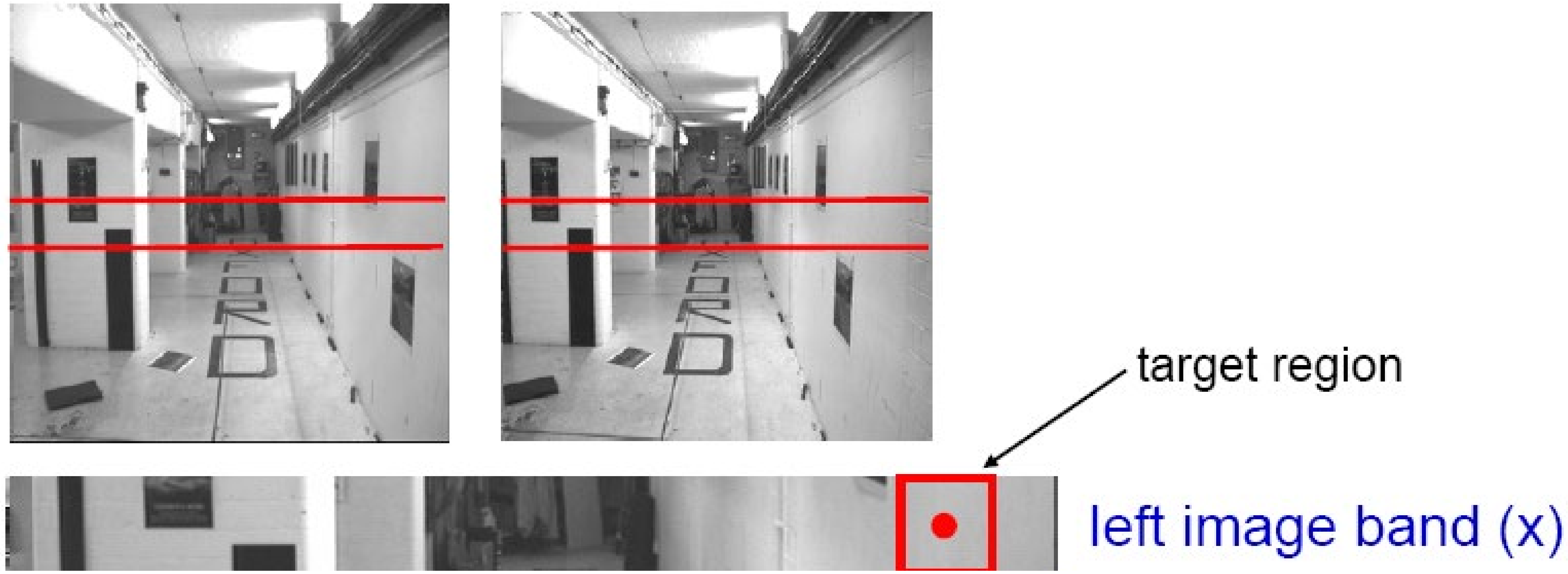
For each pixel / window in the left image

- compare with every pixel / window on same epipolar line in right image
- pick position with minimum match cost (e.g., SSD, correlation)

Adapted from Li Zhang

Grauman

Textureless regions



Source: Andrew Zisserman

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Failures of Correspondence Search

Repeated Patterns. Why?

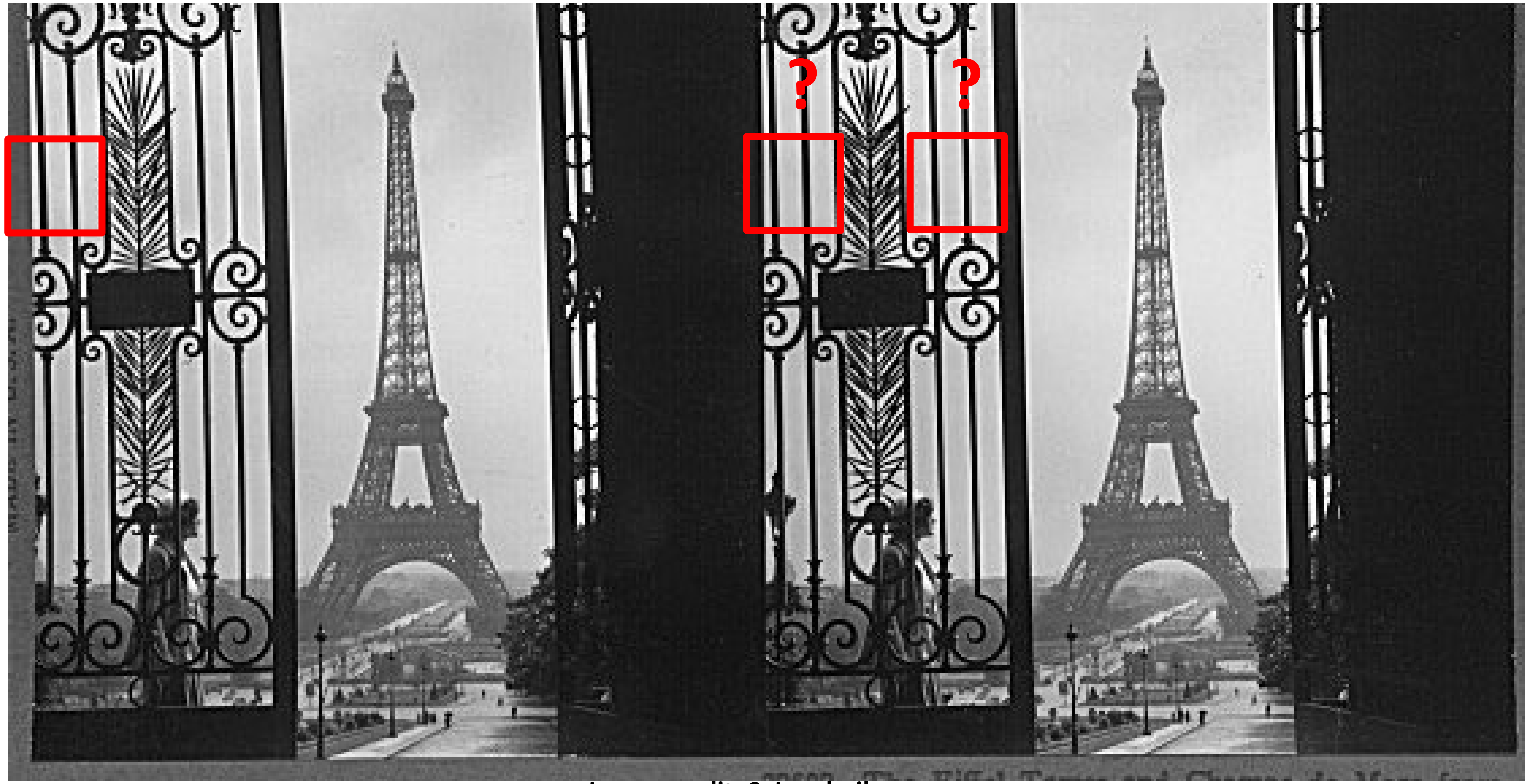
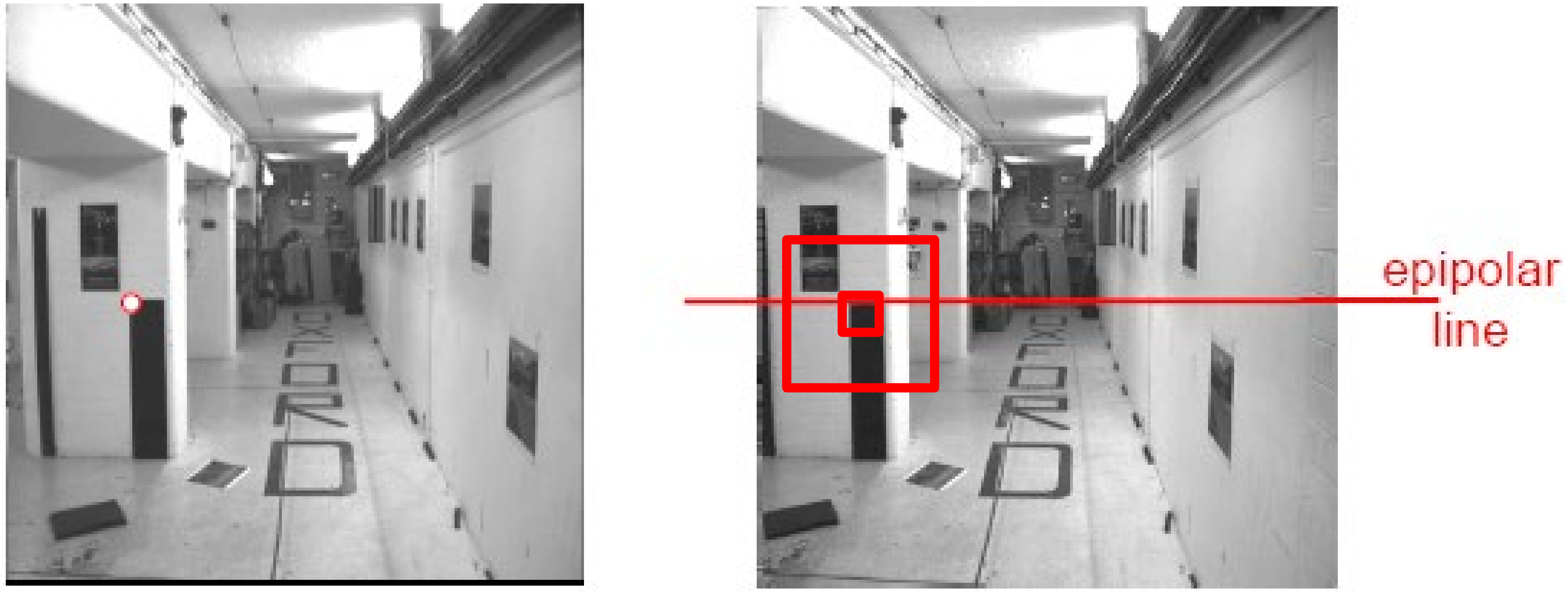


Image credit: S. Lazebnik

Effect of window size



Source: Andrew Zisserman

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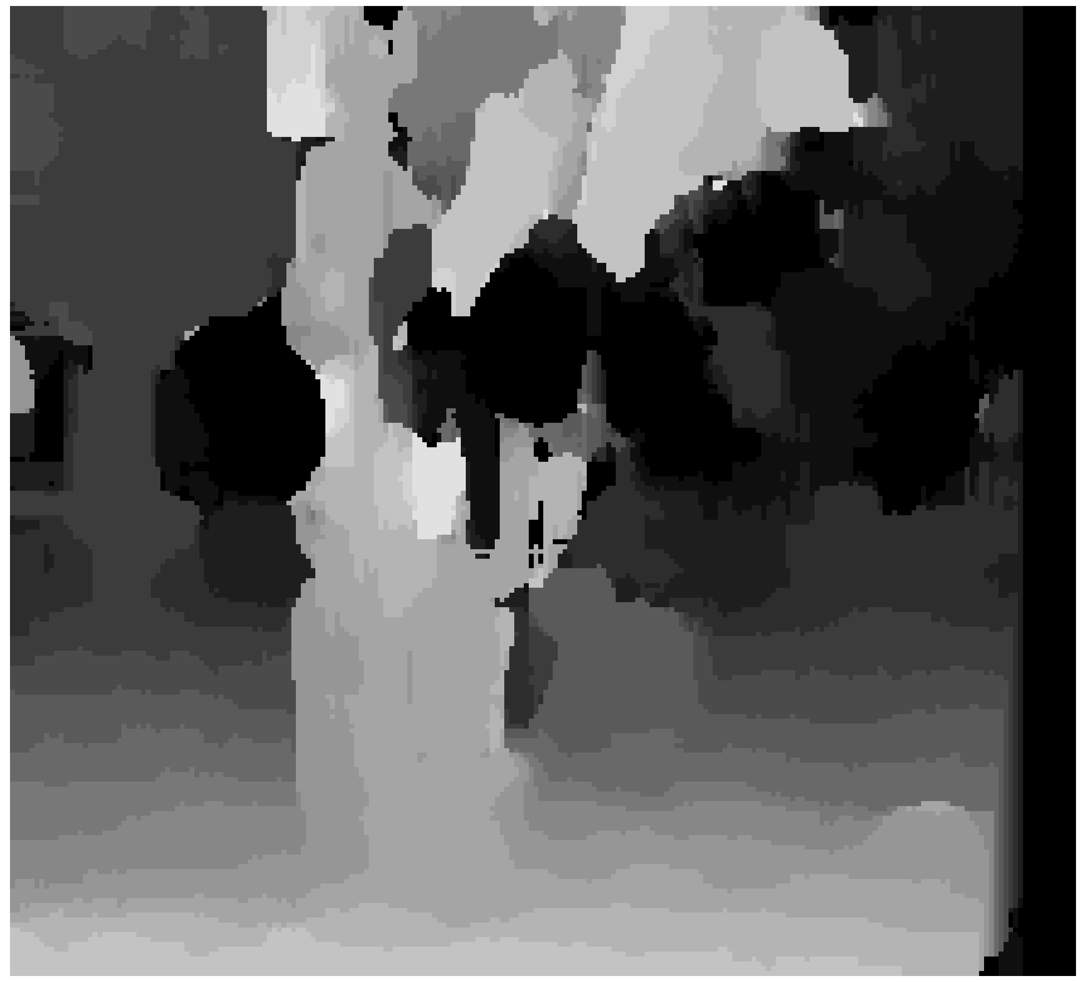
Effect of window size

W = 3



W = 20

Want window large enough to have sufficient intensity variation, yet small enough to contain only pixels with about the same disparity.

Figures from Li Zhang

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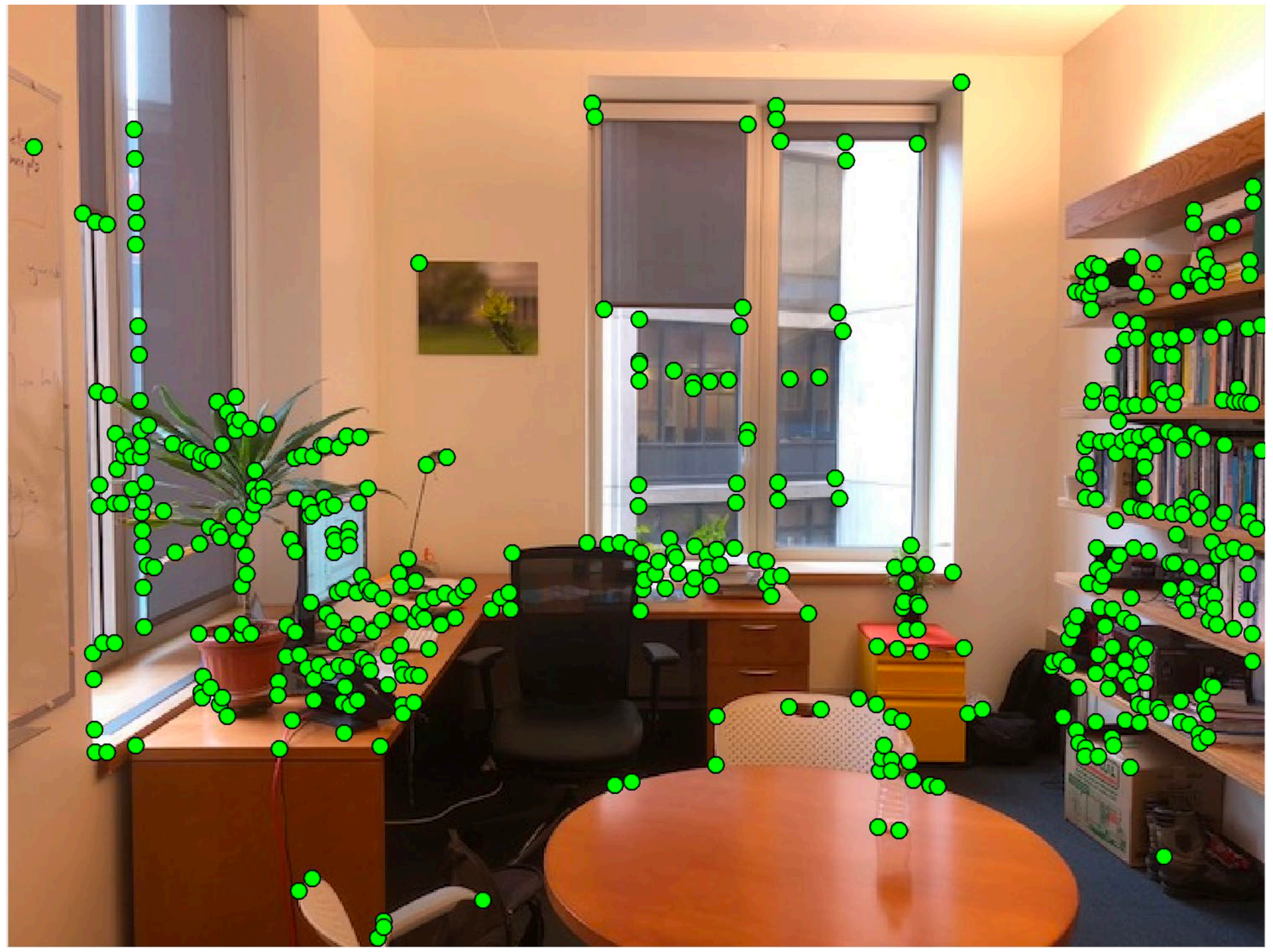
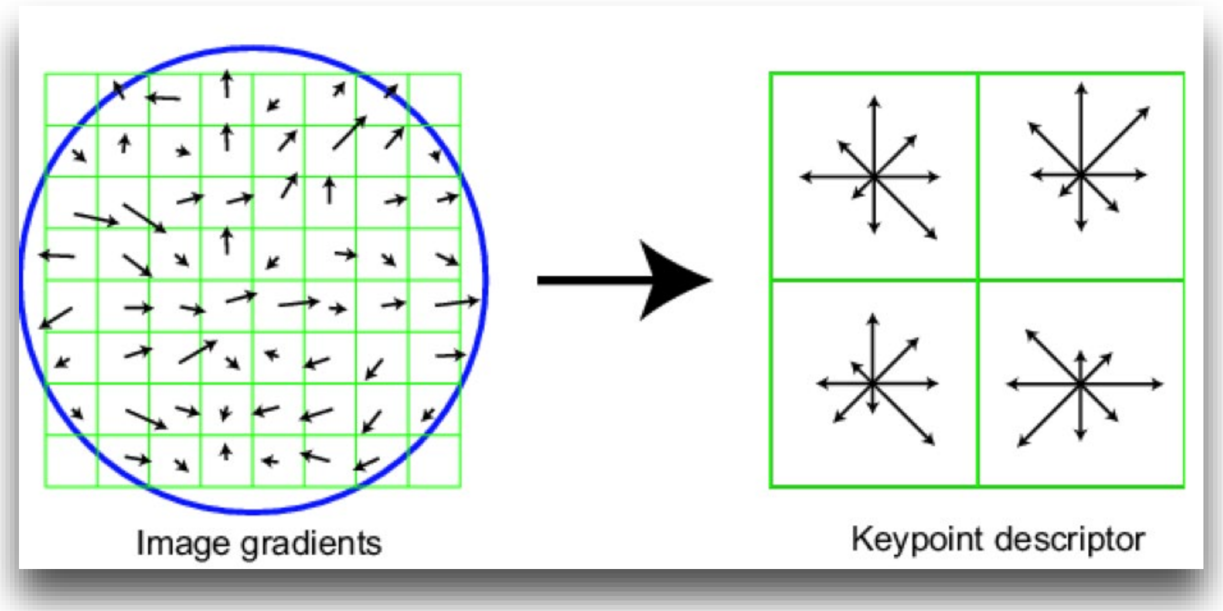
Feature correspondences

1) detect keypoints



points = detectHarrisFeatures(img);

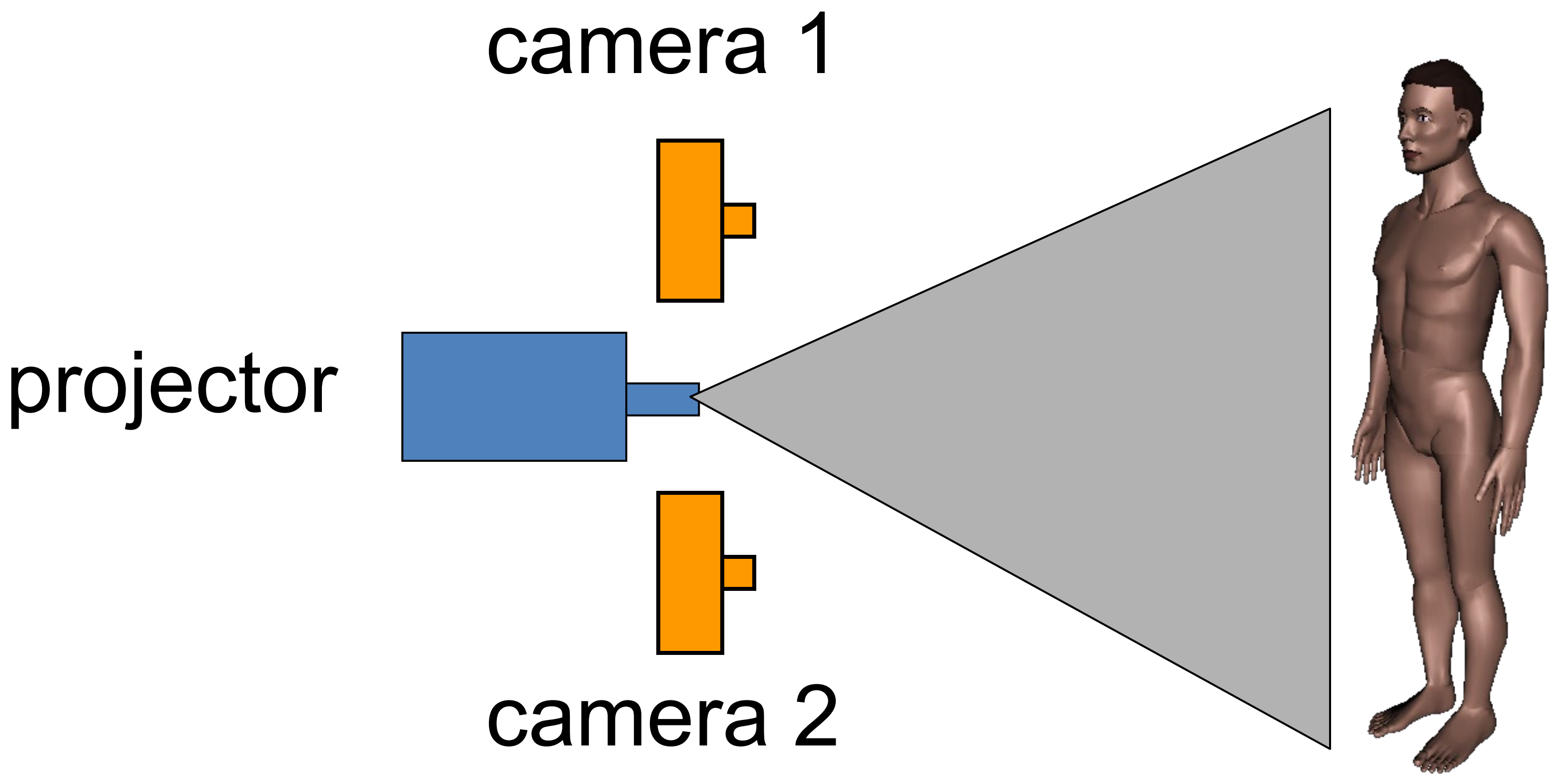
2) extract SIFT at each keypoint

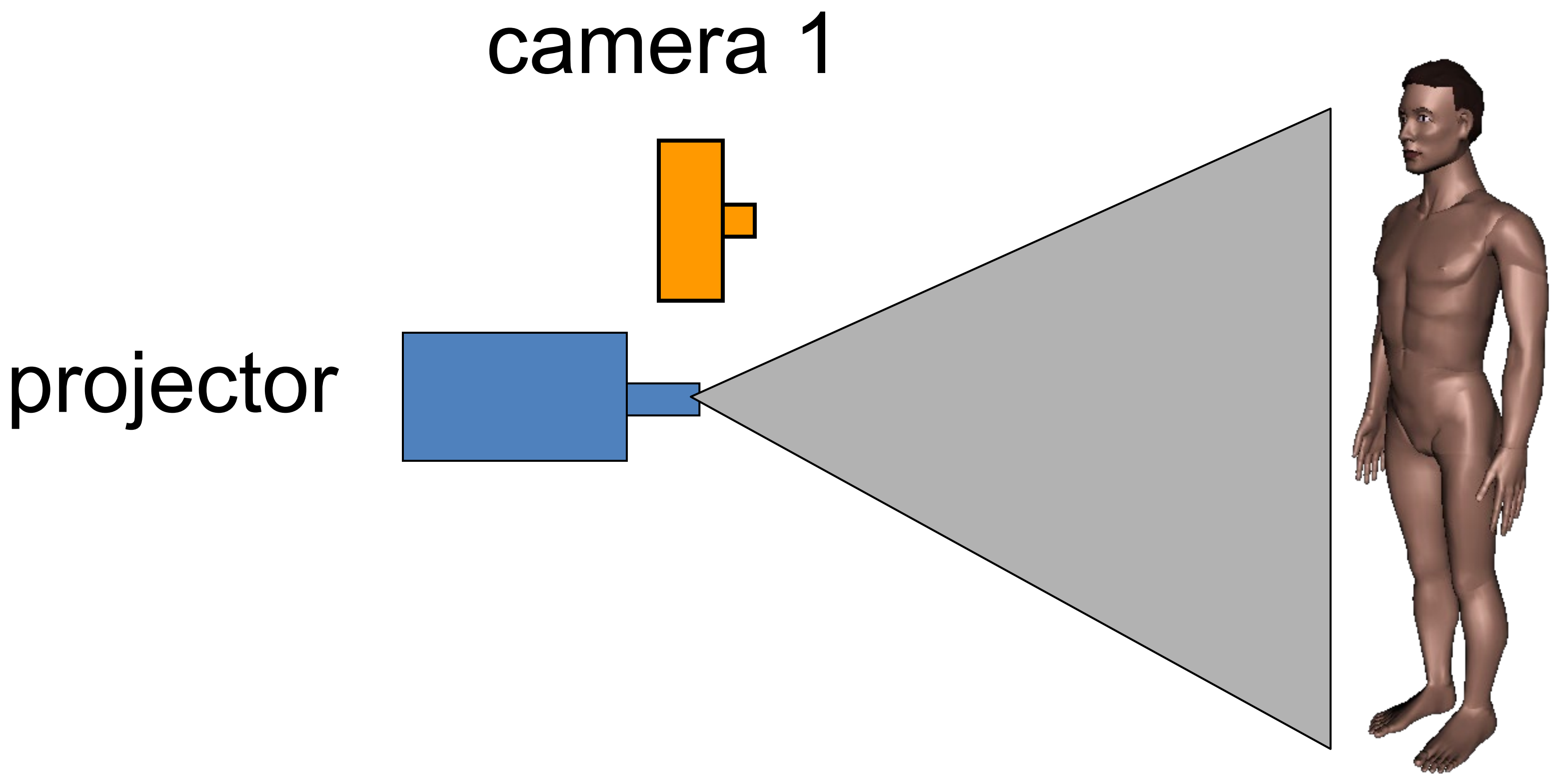


Finding correspondences (SIFT)

Active stereo with structured light

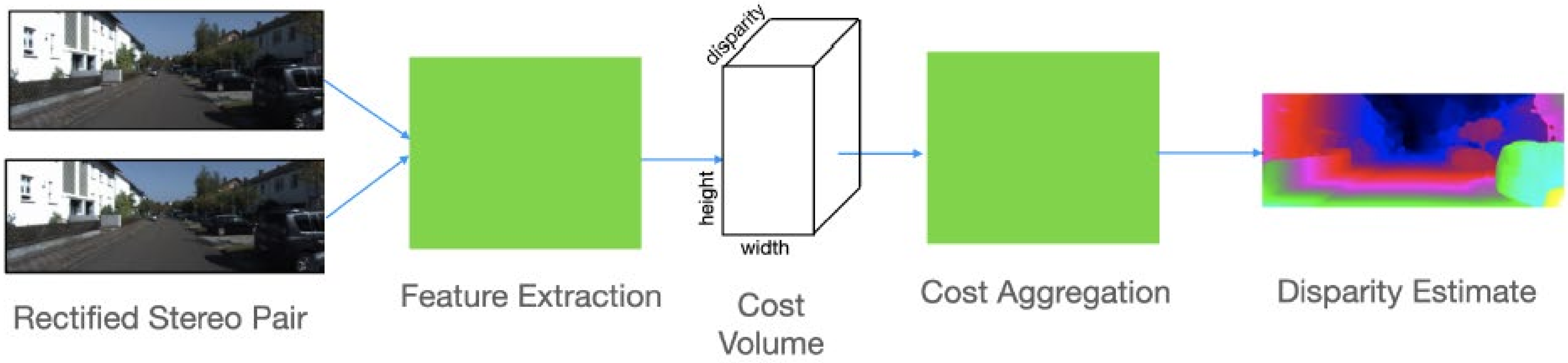
Li Zhang's one-shot stereo



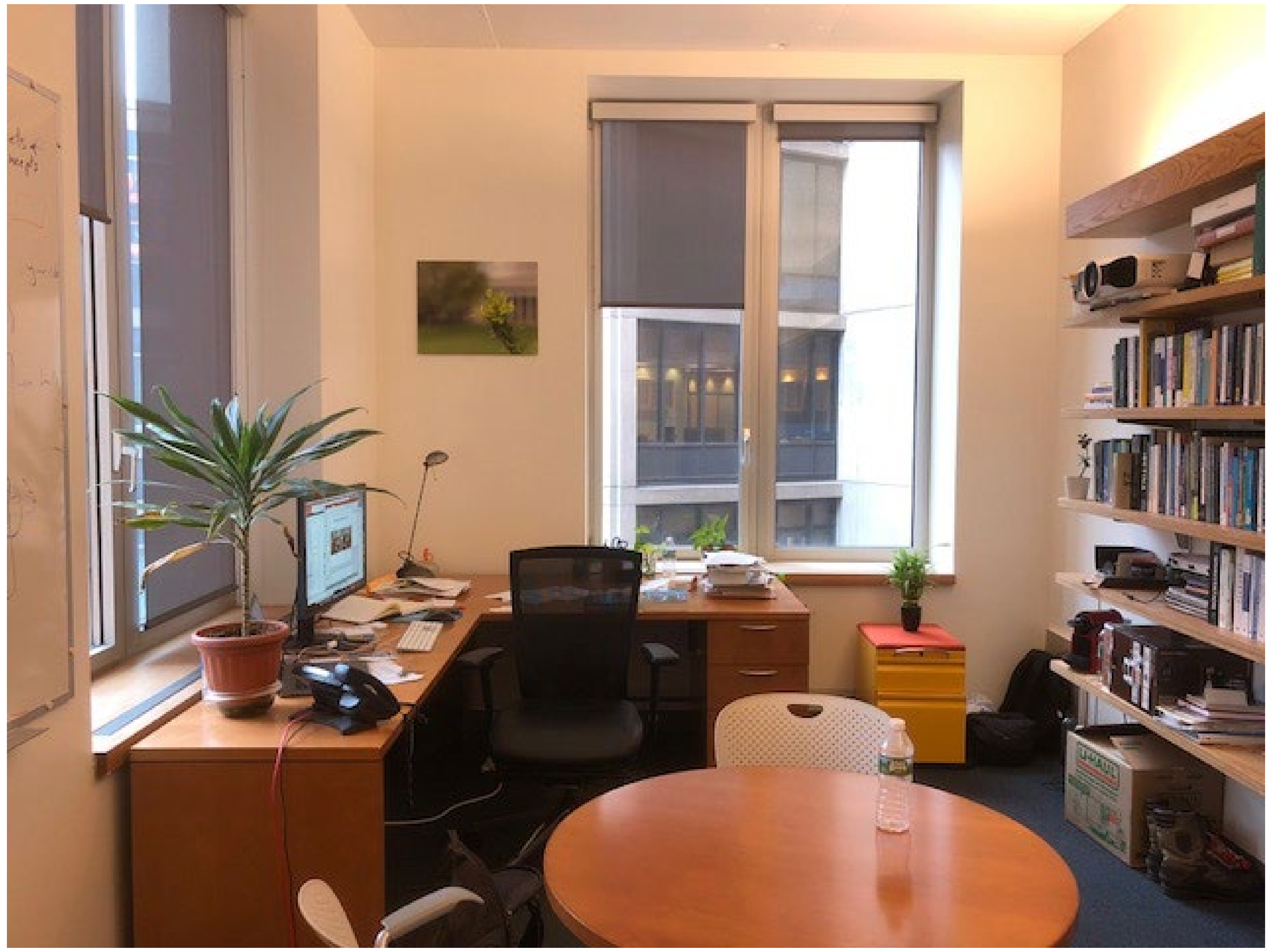


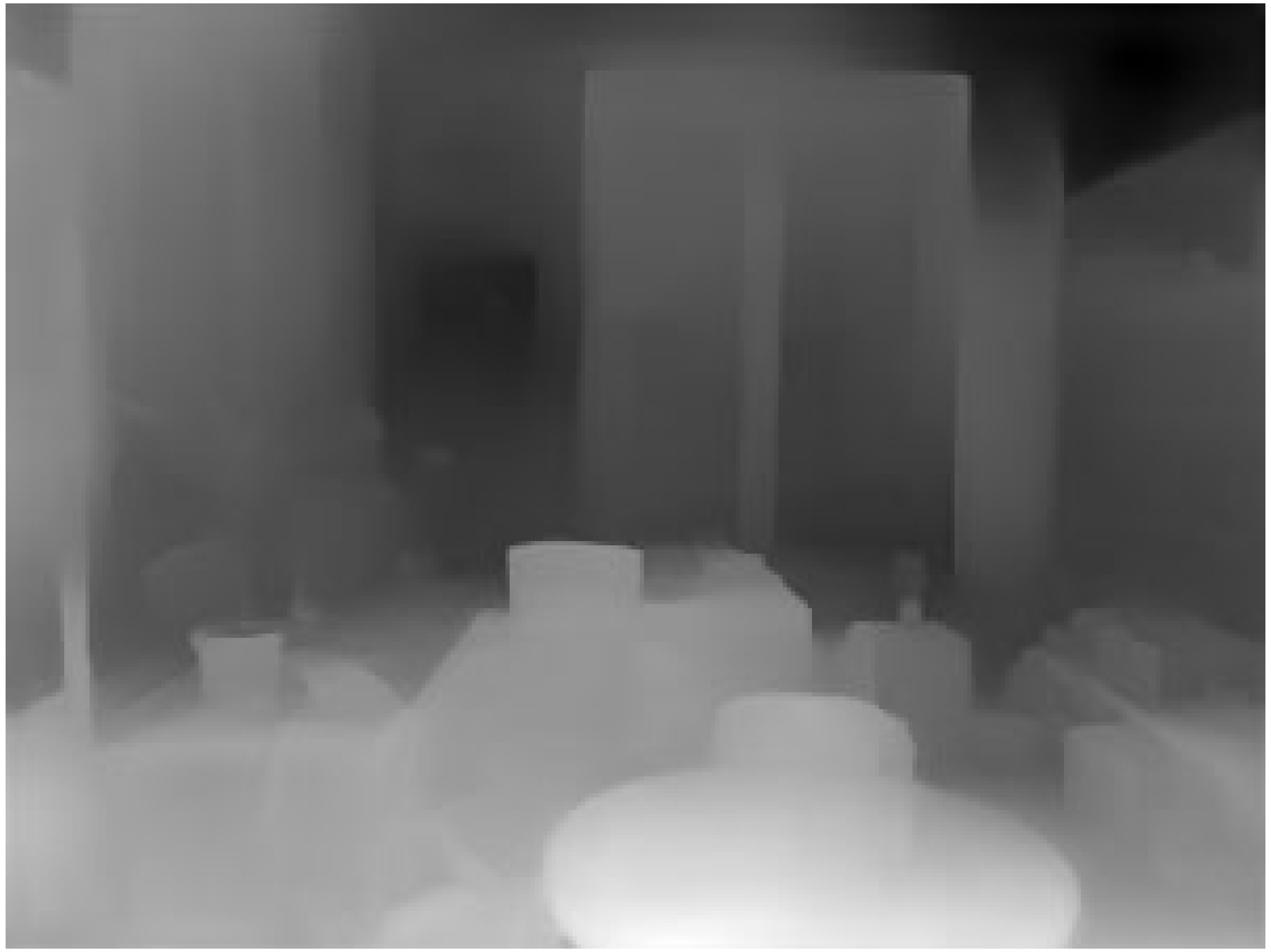
- Project "structured" light patterns onto the object
  - simplifies the correspondence problem

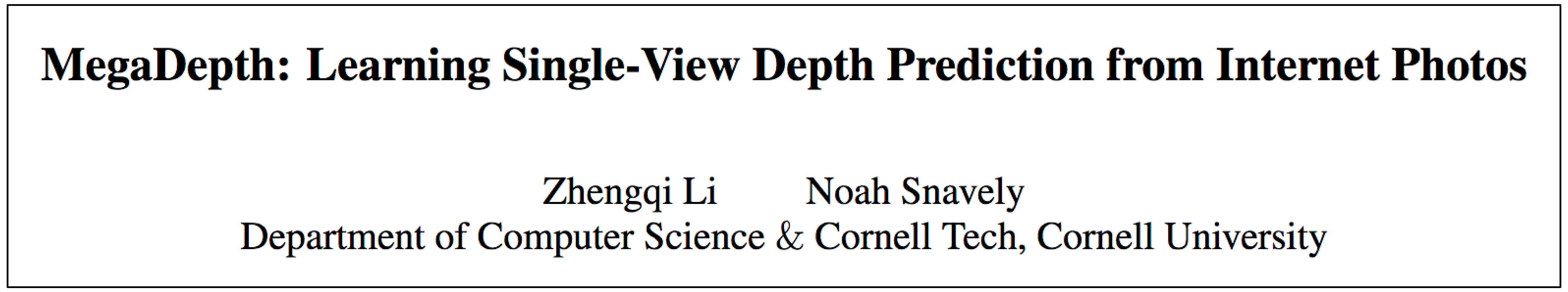
CNN-based Stereo Matching



Can also learn depth from a single image







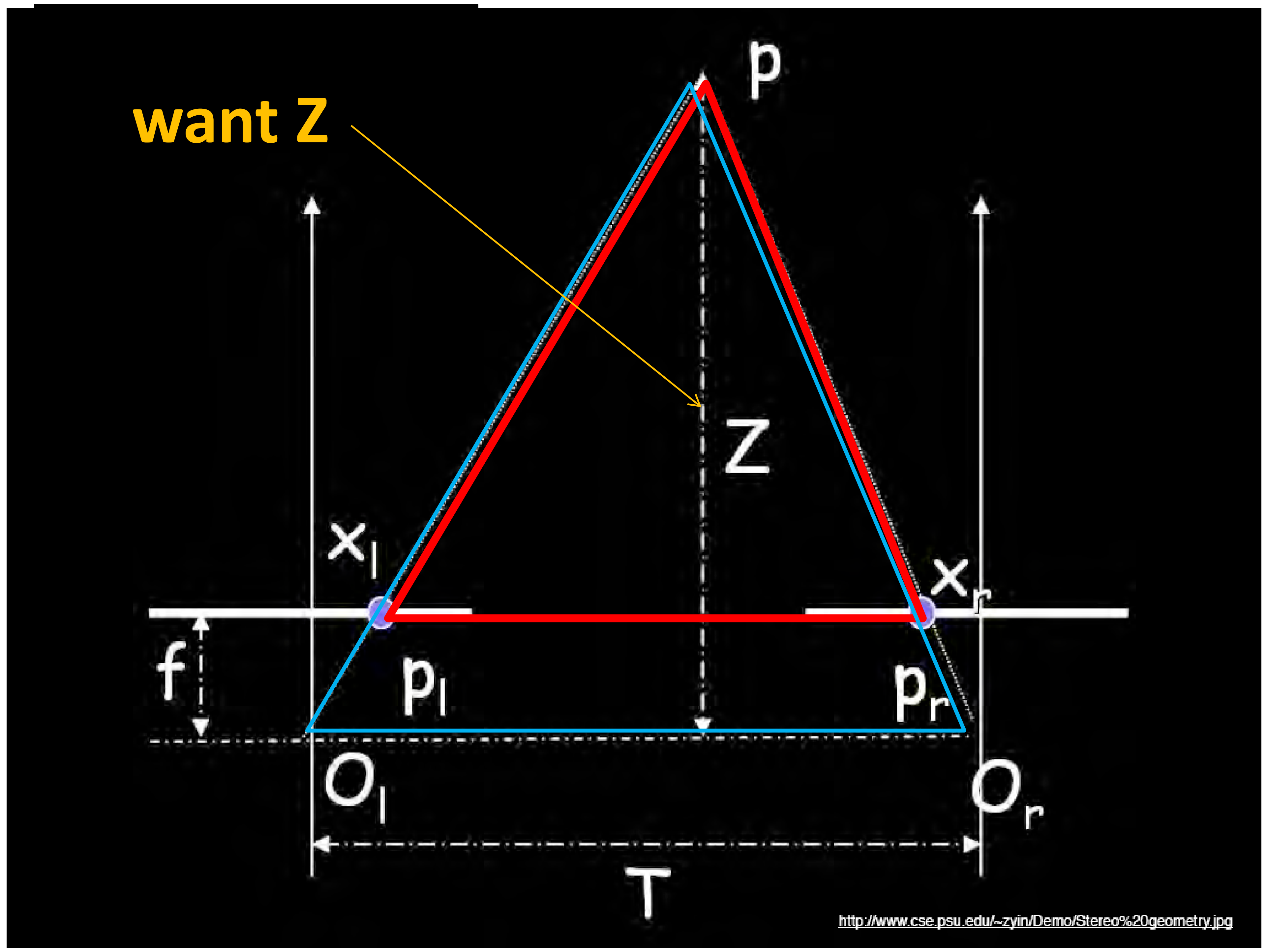
MegaDepth: Learning Single-View Depth Prediction from Internet Photos

Zhengqi Li Noah Snavely
Department of Computer Science & Cornell Tech, Cornell University

Source: Torralba, Isola, Freeman

Geometry for a simple stereo system

- Assume **parallel** optical axes, **known camera parameters (i.e., calibrated cameras).**



Use similar triangles (p_l, P, p_r) and (O_l, P, O_r):

$$\frac{T + x_l - x_r}{Z - f} = \frac{T}{Z}$$

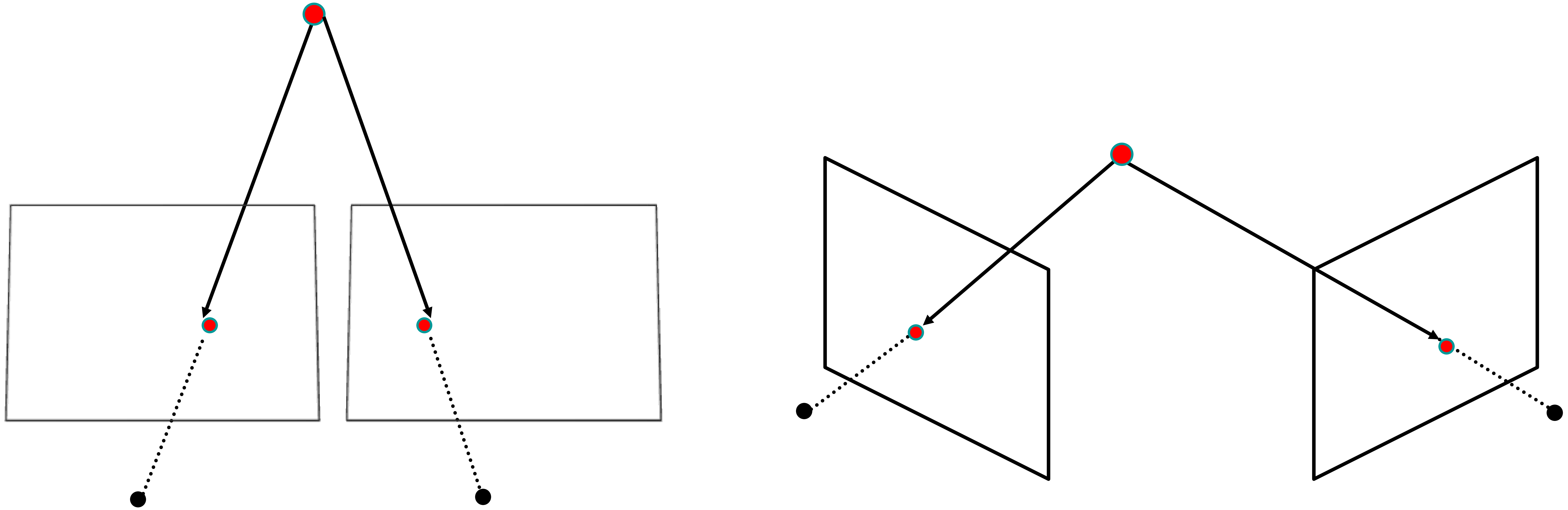
$$Z = f \frac{T}{x_r - x_l}$$

disparity

Grauman

General case

- The two cameras need not have parallel optical axes.

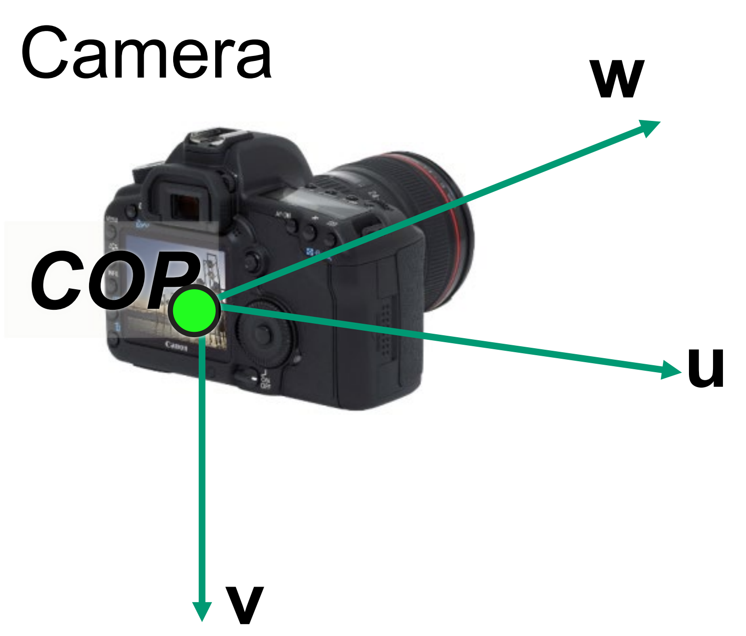


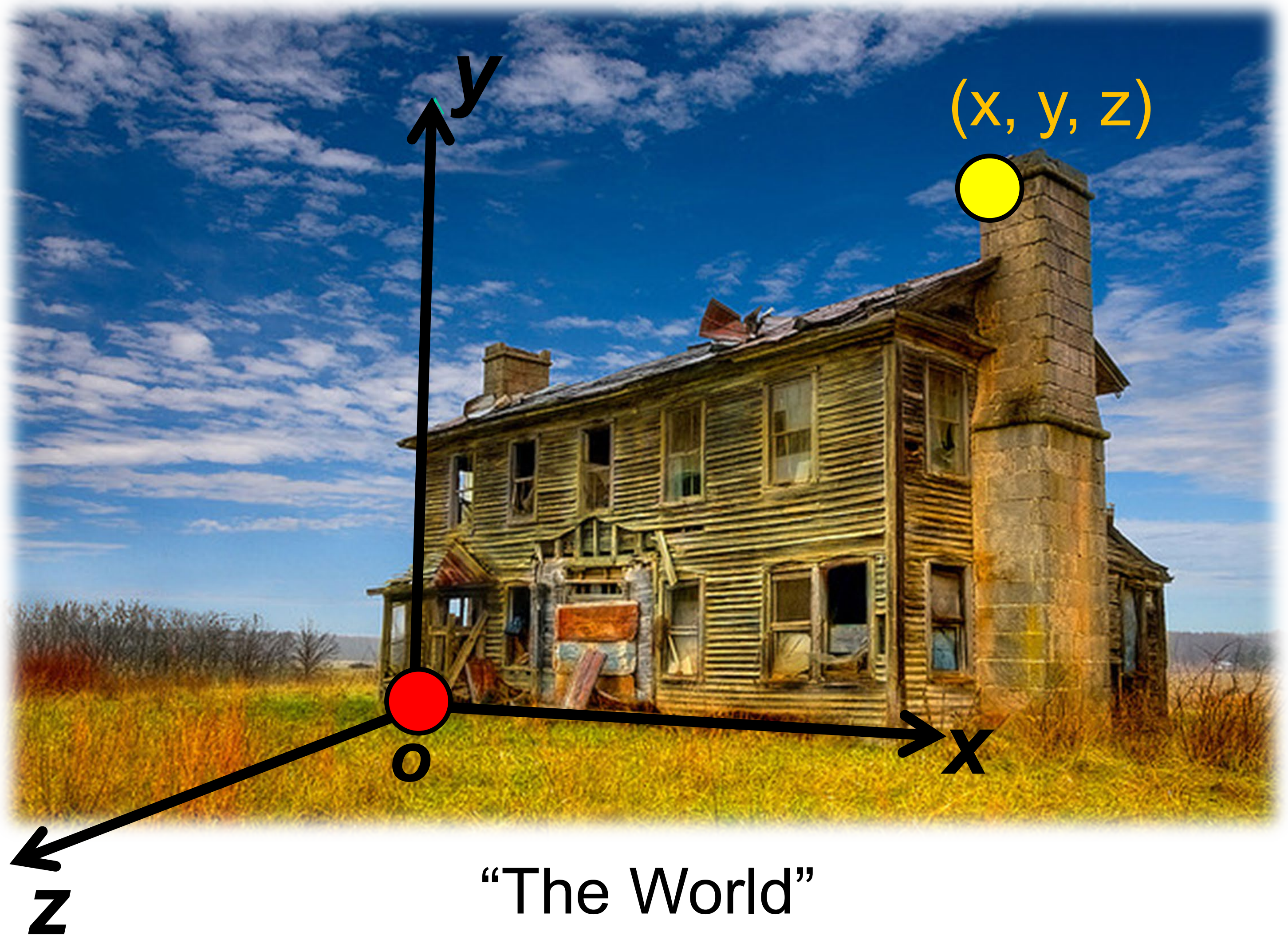
Source: Torralba, Isola, Freeman

Situating Camera in the world

There is a world coordinate frame and camera looking at the world

How can we model the geometry of a camera?





"The World"

Three important coordinate systems:

1. *World* coordinates
2. *Camera* coordinates
3. *Image* coordinates

Slide credit: Noah Snavely

Coordinate frames + Transforms

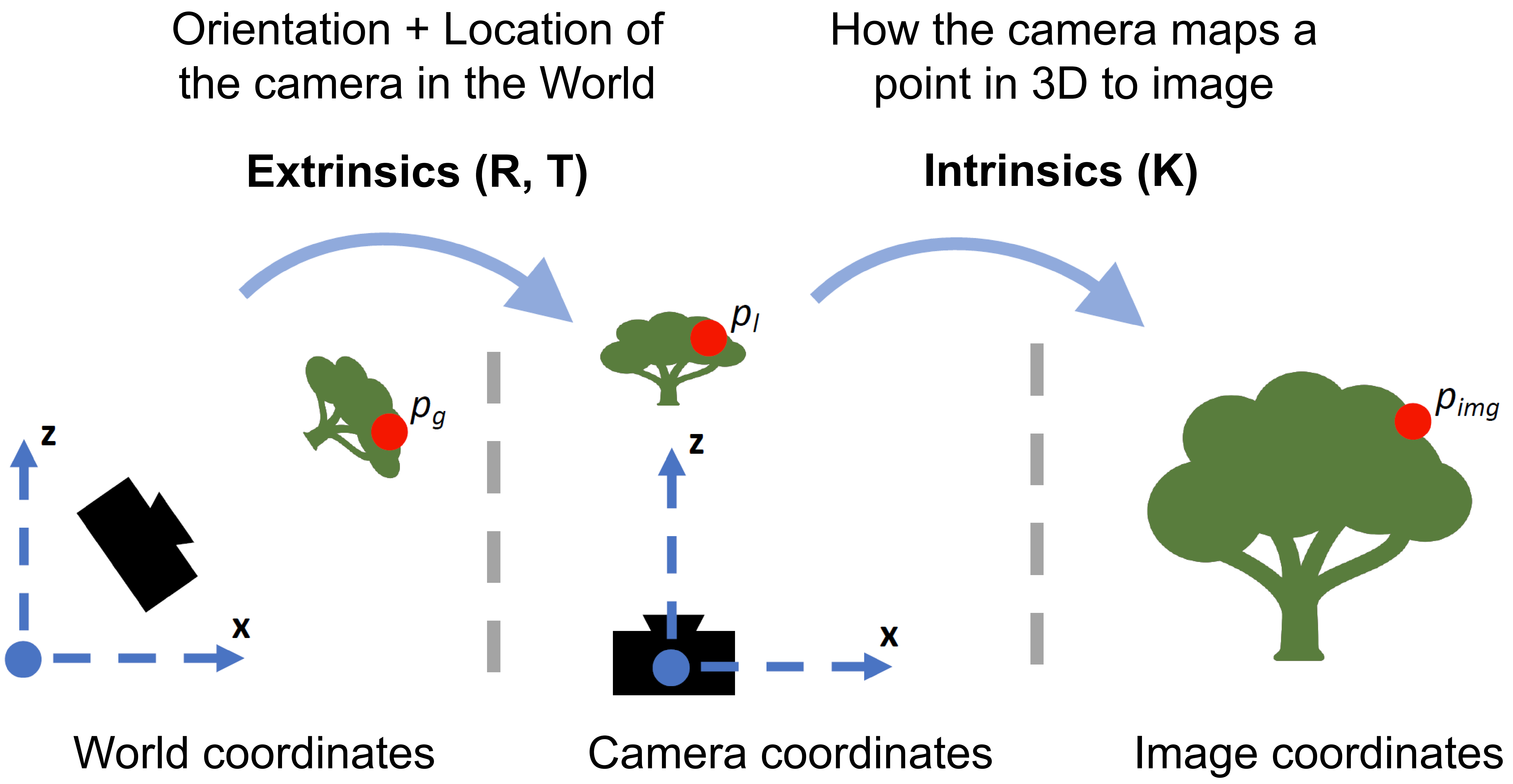


Figure credit: Peter Hedman

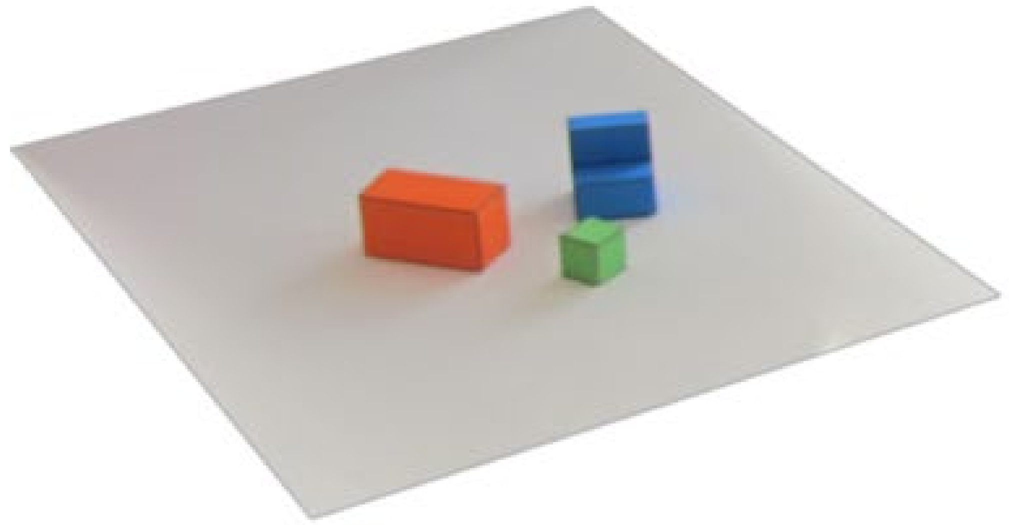
Review: Camera parameters

(x, y) 2D image

?

3D world

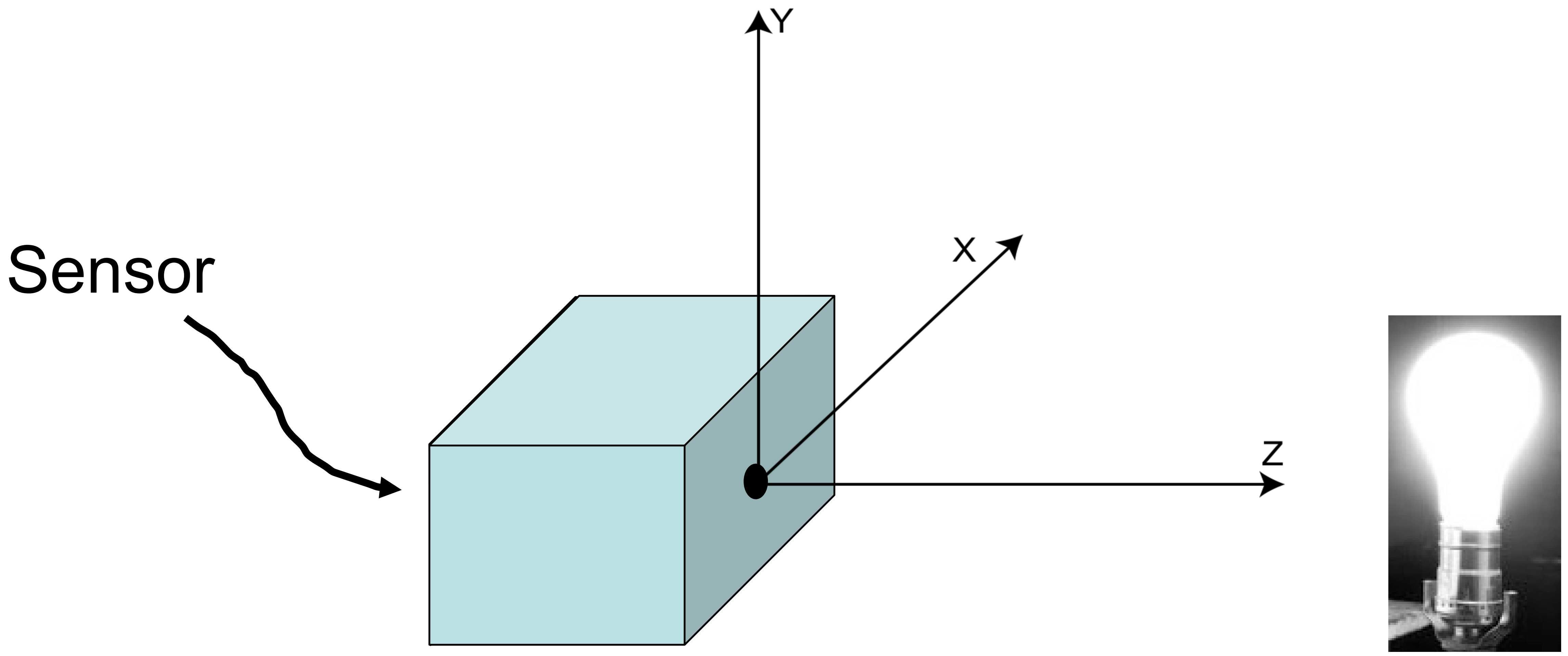
(X, Y, Z)



$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Pixels **World coordinates**

Camera parameters

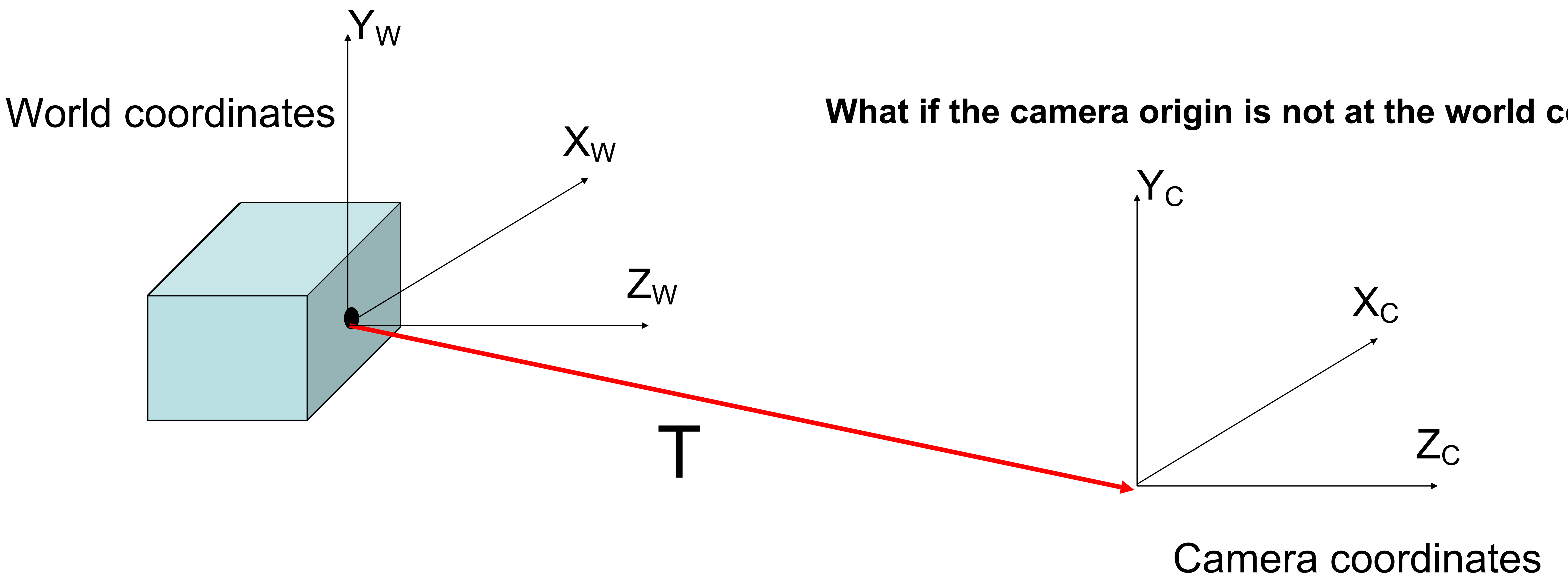


if pixels are rectangular

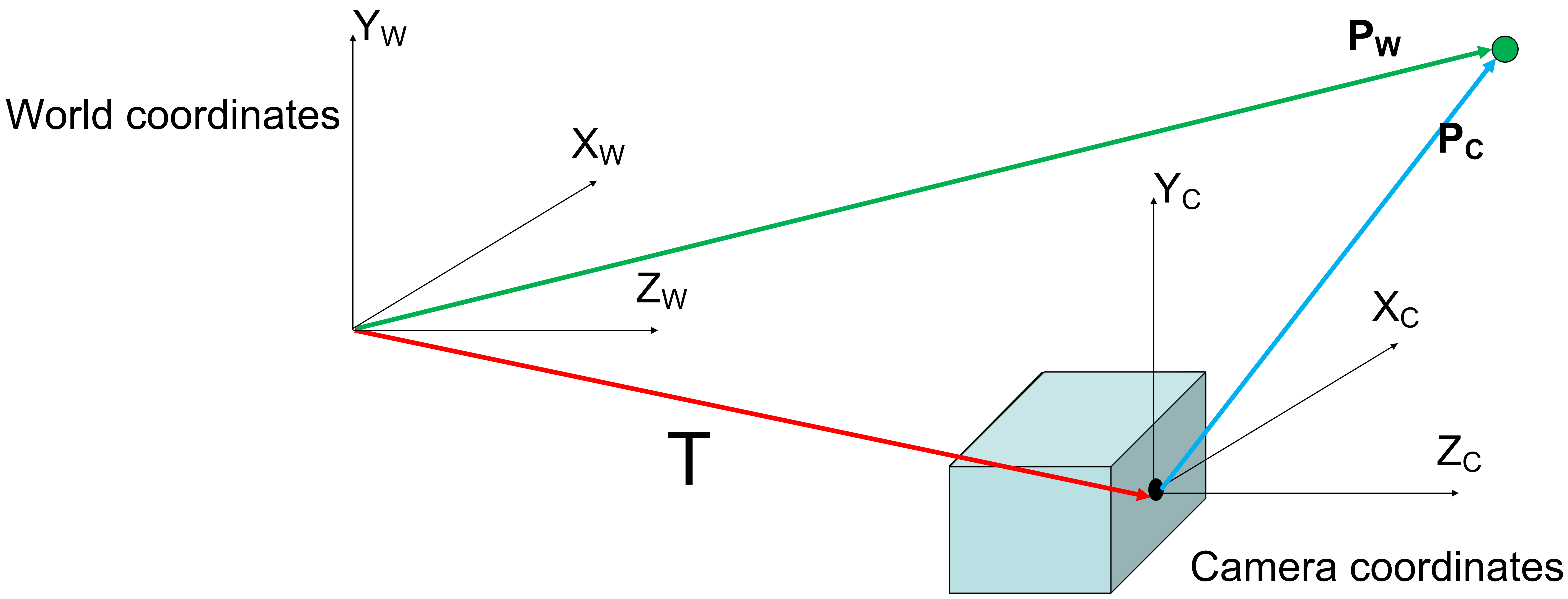
$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} aX \\ bY \\ Z \end{bmatrix} \longrightarrow (a\,X/Z,\ b\,Y/Z)$$

Camera parameters

What if the camera origin is not at the world coordinates origin?



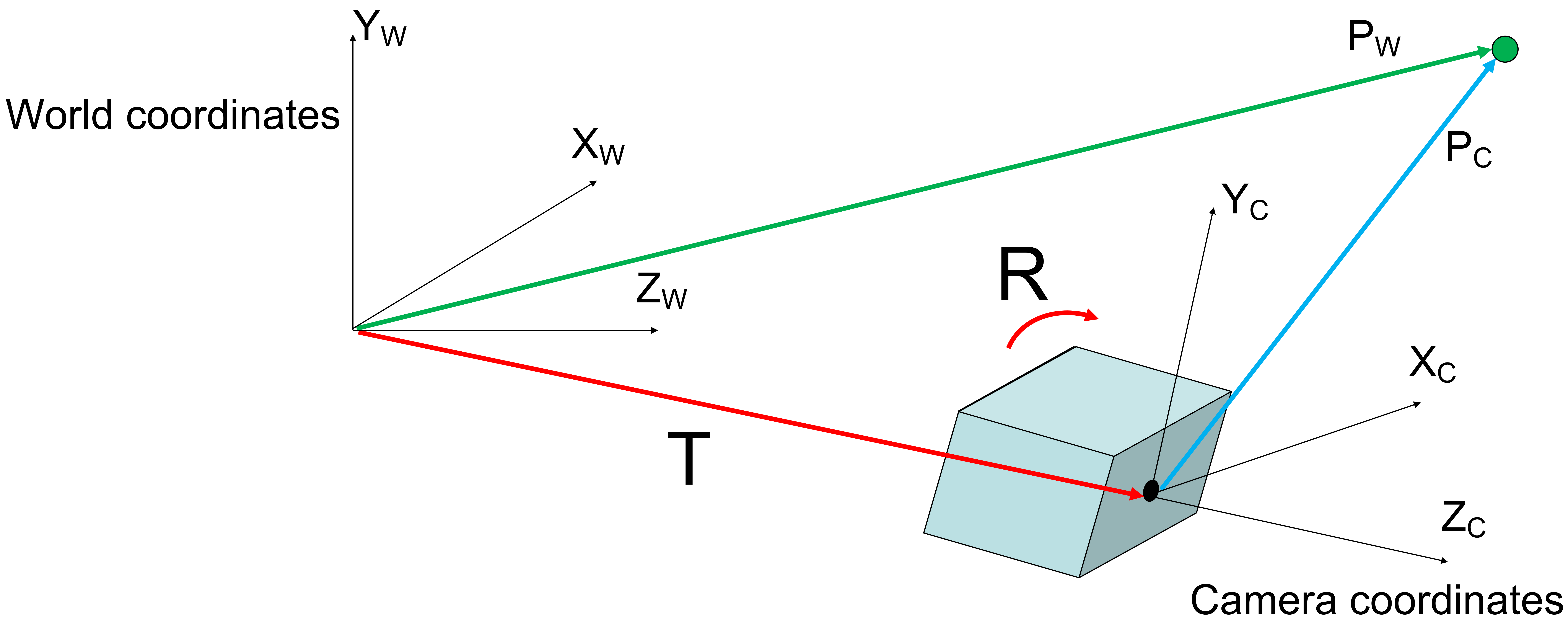
Camera parameters



In heterogeneous coordinates:

$$P_C = P_W - T$$

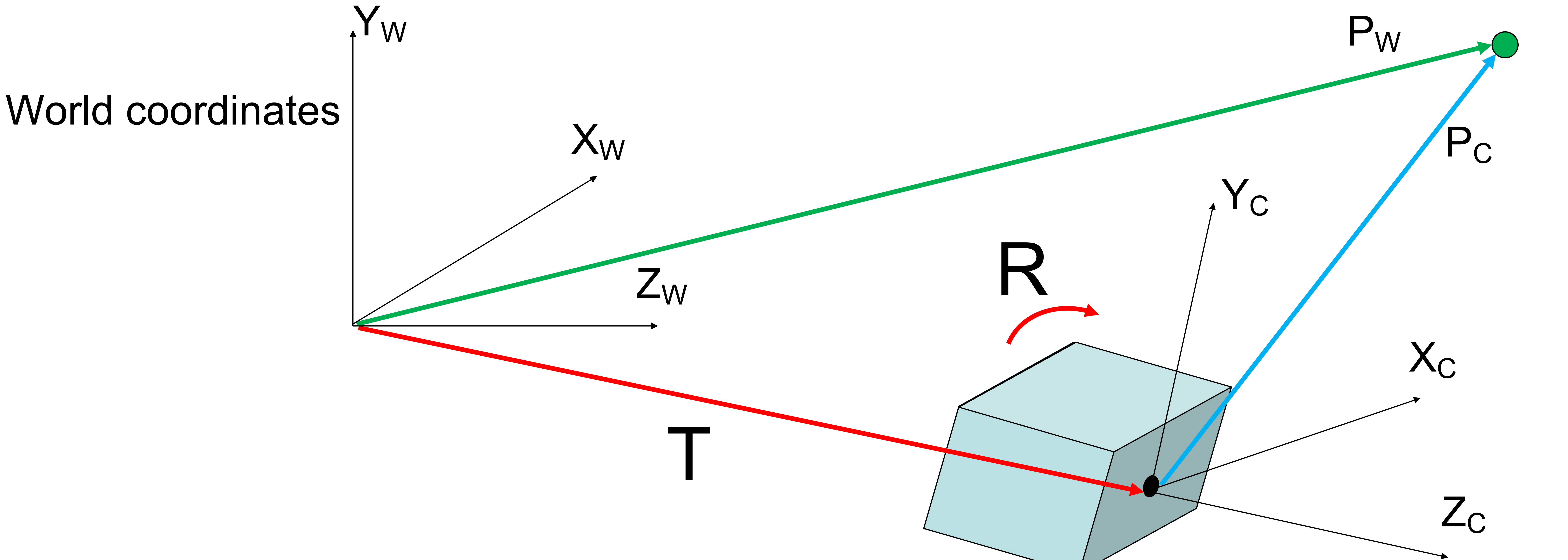
Camera parameters



In heterogeneous coordinates:

$$P_C = R(P_W - T)$$

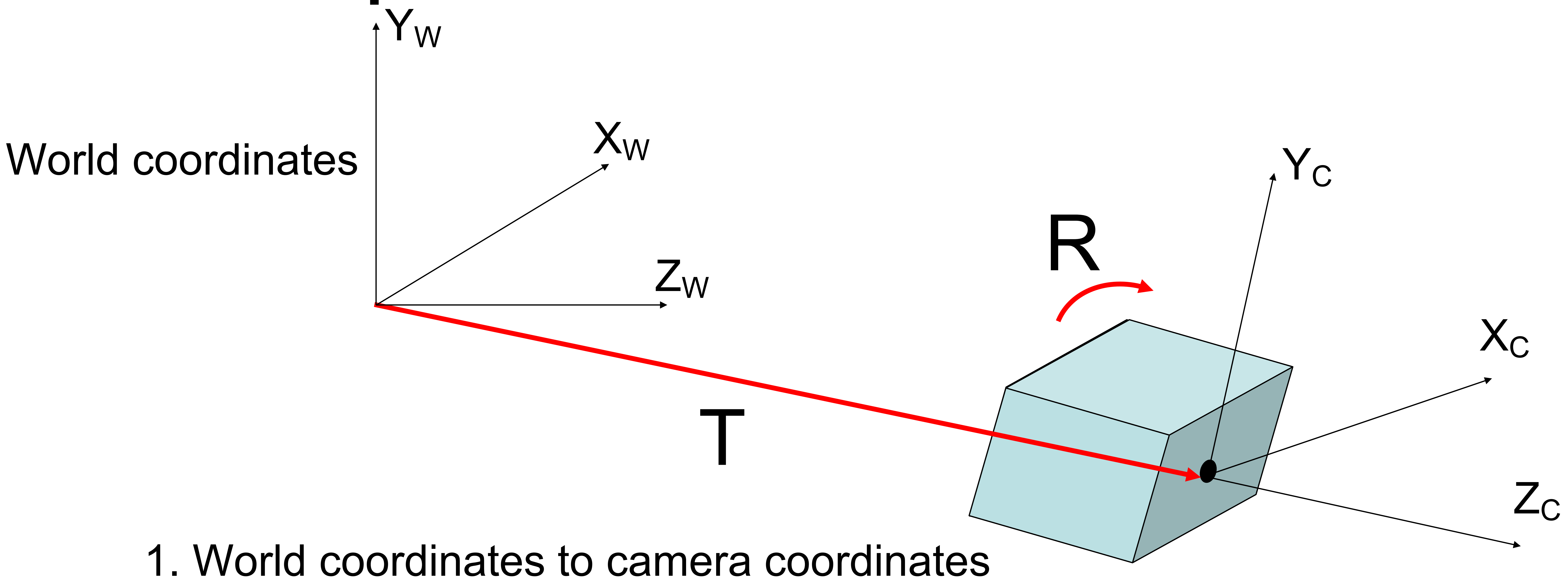
Camera parameters



In homogeneous coordinates:

$$\begin{bmatrix}X_c\\Y_c\\Z_c\\1\end{bmatrix} = \begin{bmatrix} \overset{[3\times3]}{\mathbf{R}} & \overset{[3\times1]}{\mathbf{-RT}} \\ \underset{[1\times3]}{\mathbf{0}} & \underset{[1\times1]}{1}\end{bmatrix} \cdot \begin{bmatrix}X_w\\Y_w\\Z_w\\1\end{bmatrix}$$

Camera parameters



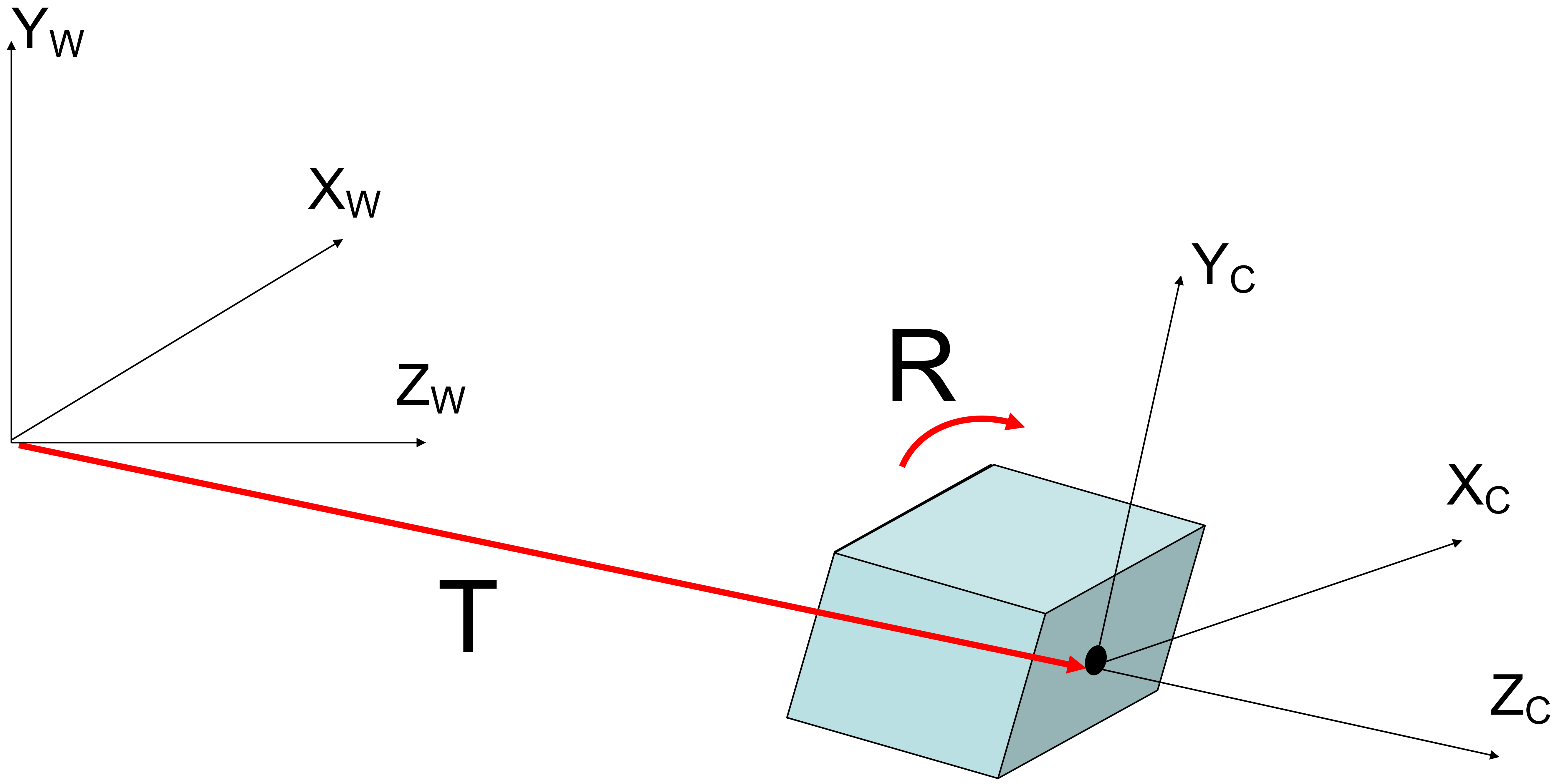
1. World coordinates to camera coordinates

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{-RT} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

2. Camera coordinates to image coordinates (square pixels)

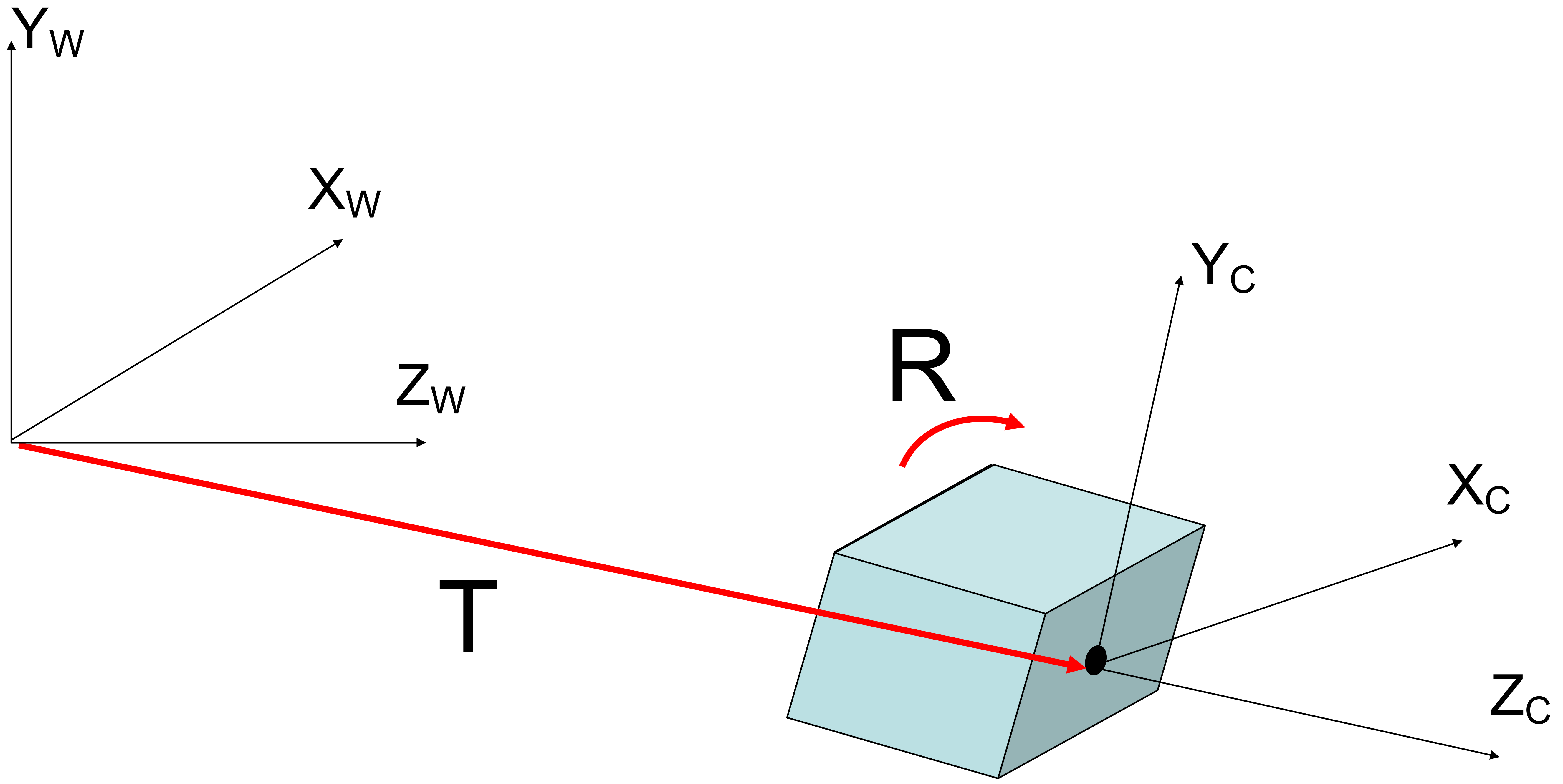
$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

Camera parameters



$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

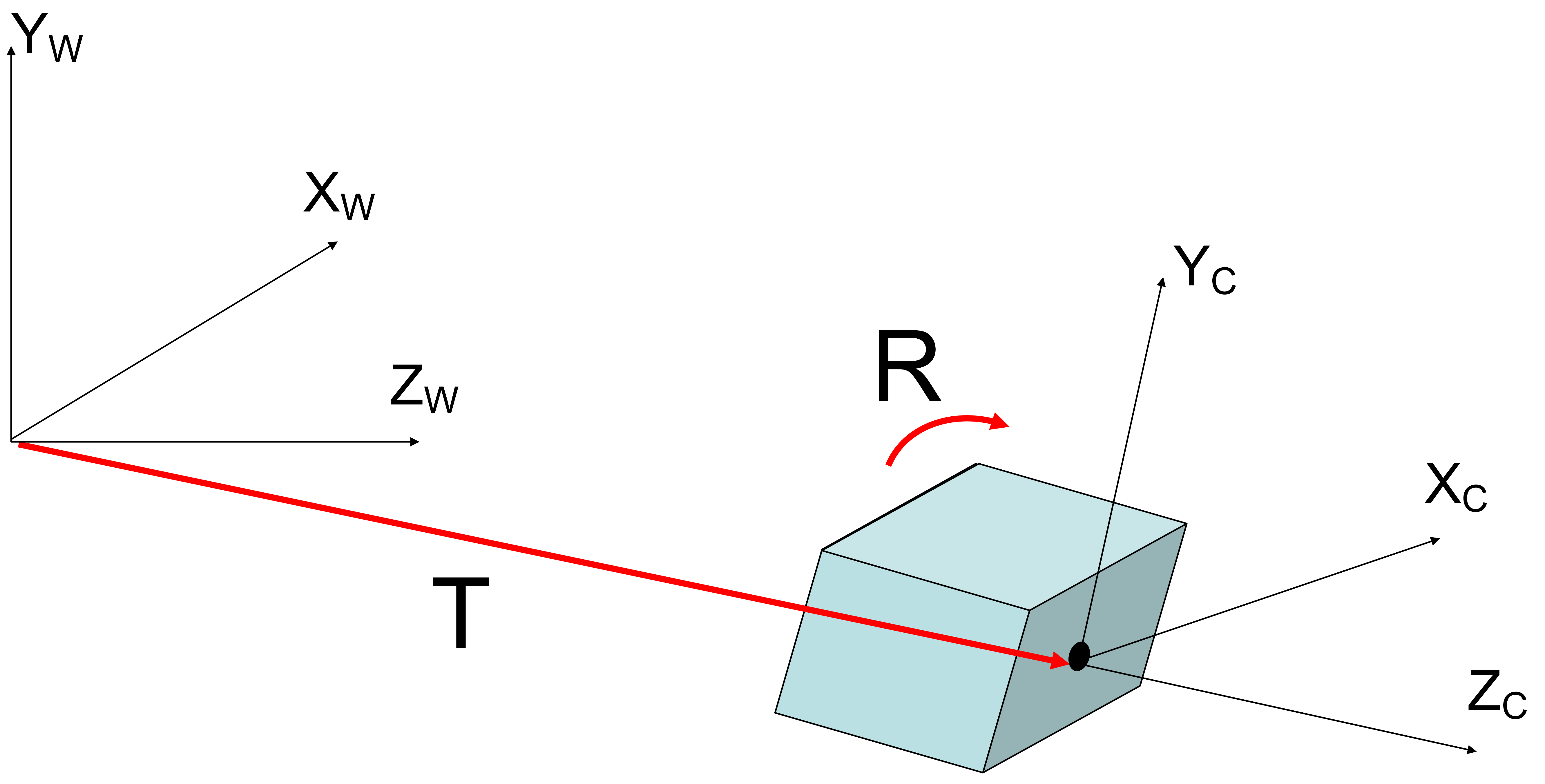
Camera parameters



[3x4] [4x4]

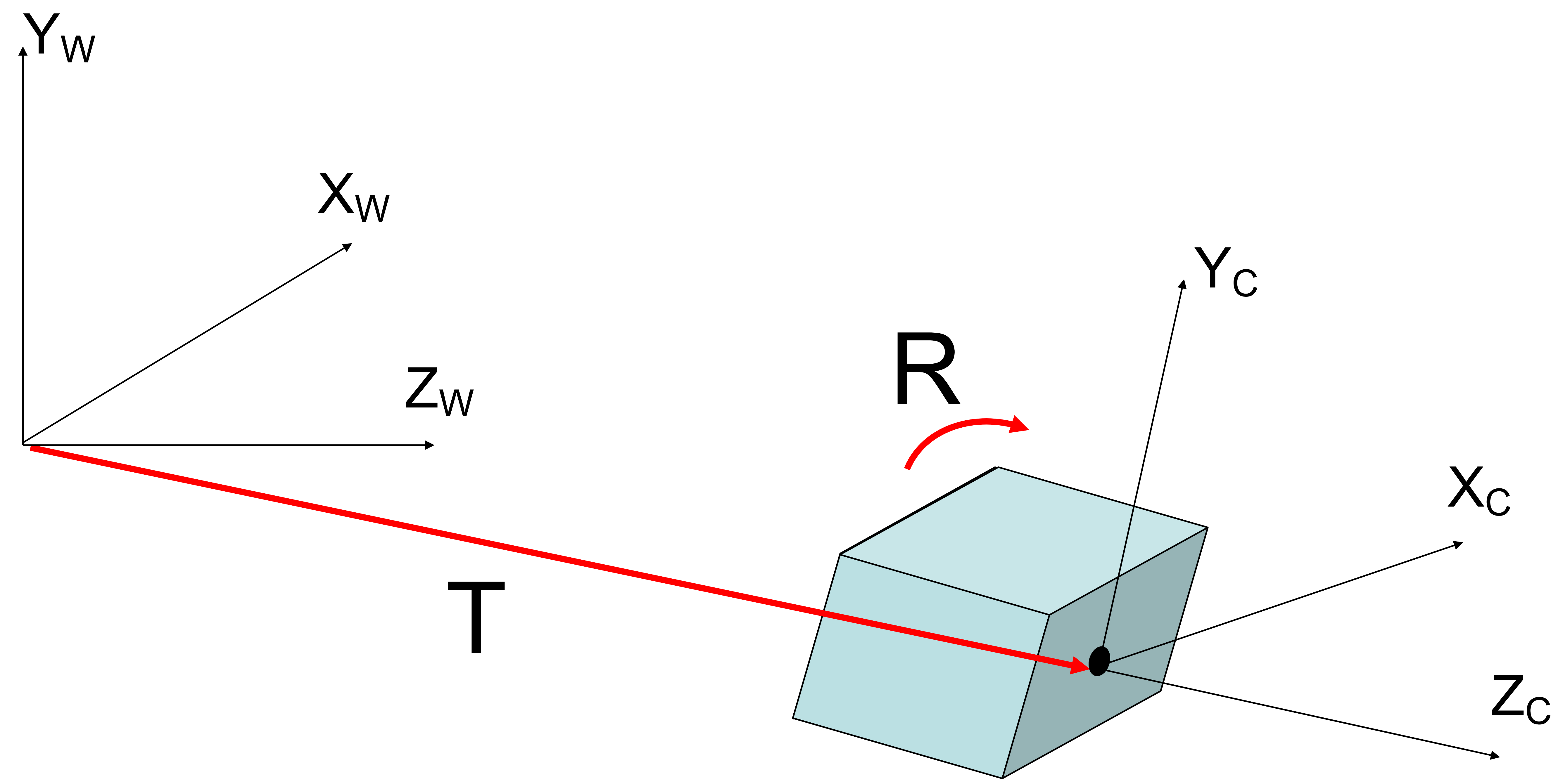
$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{R} & \mathbf{-RT} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Camera parameters



$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[3x3]} \cdot \underbrace{\begin{bmatrix} \mathbf{R} & \mathbf{-RT} \end{bmatrix}}_{[3x4]} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Camera parameters



$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \underbrace{\overset{[3x3]}{\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}}}_{\text{Intrinsic parameters}} \cdot \underbrace{\overset{[3x3]}{\begin{bmatrix} \mathbf{R} \end{bmatrix}} \cdot \overset{[3x4]}{\begin{bmatrix} \mathbf{I} & \mathbf{-T} \end{bmatrix}}}_{\text{Extrinsic parameters}} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Camera Projection Model

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

5 6

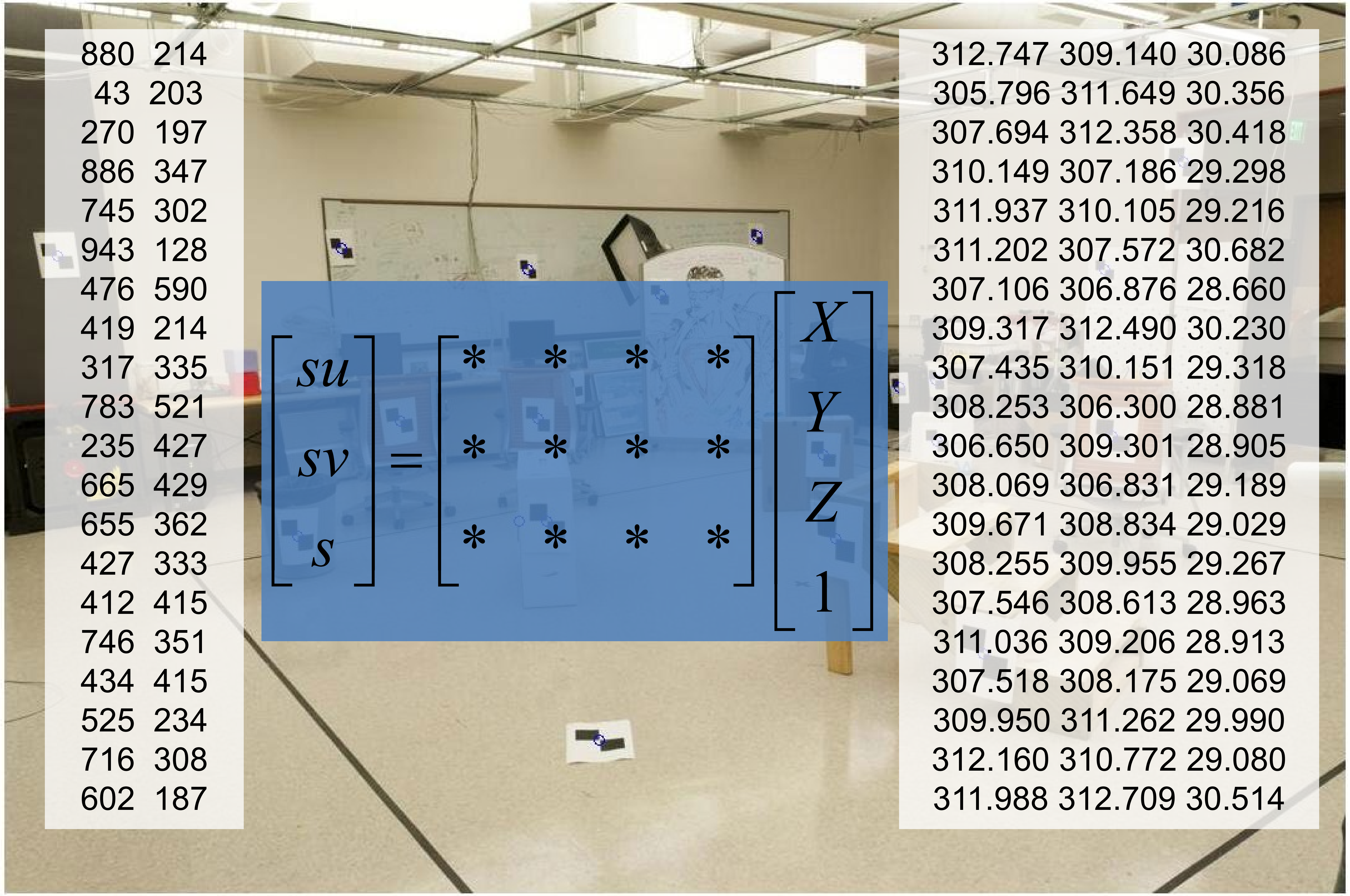
$$w\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

How to calibrate the camera?

$$\mathbf{x} = \mathbf{K}\begin{bmatrix}\mathbf{R} & \mathbf{t}\end{bmatrix}\mathbf{X}$$

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we calibrate a camera? Learning problem!



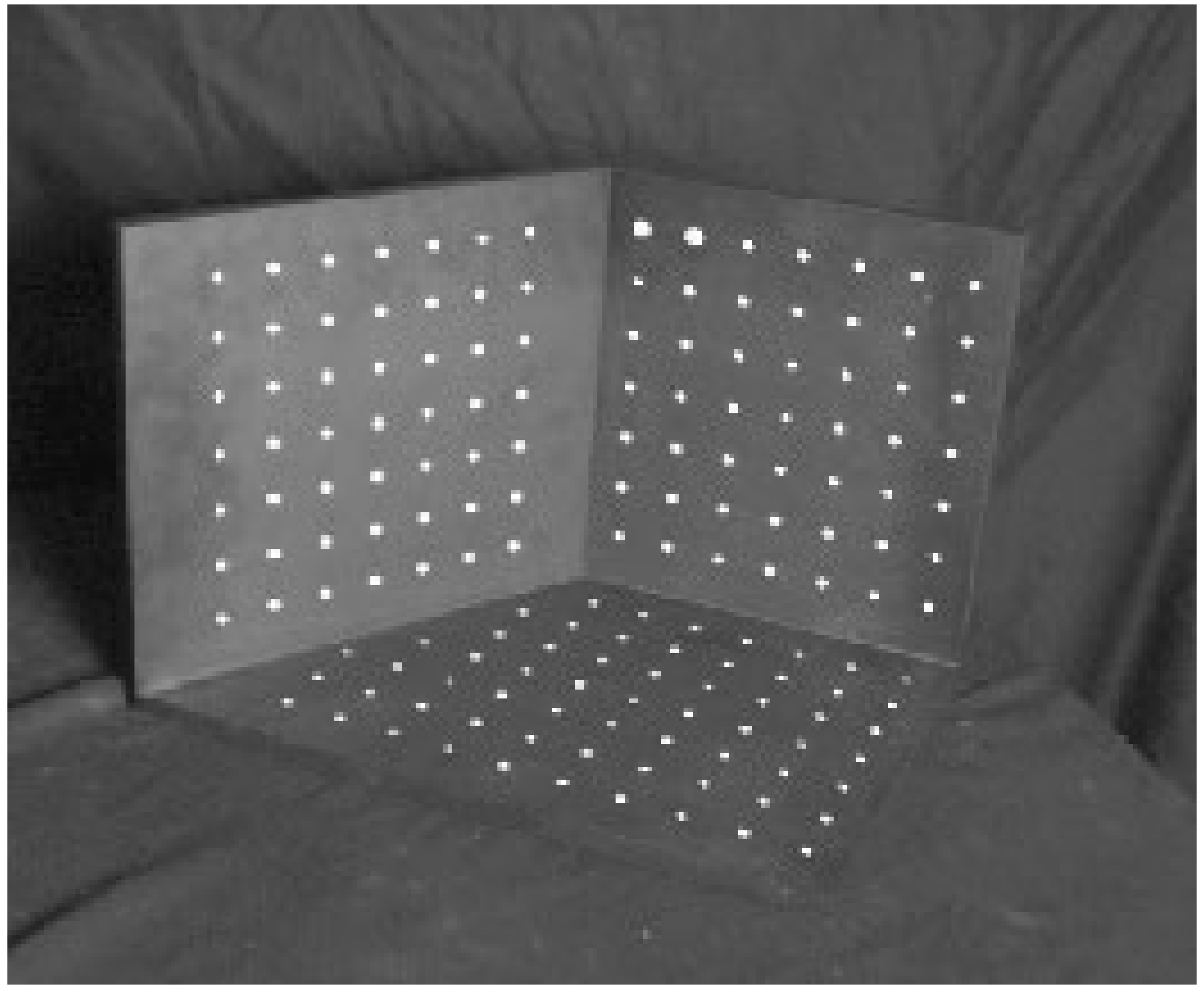
$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

880	214
43	203
270	197
886	347
745	302
943	128
476	590
419	214
317	335
783	521
235	427
665	429
655	362
427	333
412	415
746	351
434	415
525	234
716	308
602	187

312.747	309.140	30.086
305.796	311.649	30.356
307.694	312.358	30.418
310.149	307.186	29.298
311.937	310.105	29.216
311.202	307.572	30.682
307.106	306.876	28.660
309.317	312.490	30.230
307.435	310.151	29.318
308.253	306.300	28.881
306.650	309.301	28.905
308.069	306.831	29.189
309.671	308.834	29.029
308.255	309.955	29.267
307.546	308.613	28.963
311.036	309.206	28.913
307.518	308.175	29.069
309.950	311.262	29.990
312.160	310.772	29.080
311.988	312.709	30.514

Calibration using a reference object

- Place a known object in the scene
  - identify correspondence between image and scene
  - compute mapping from scene to image



Issues

- **must know geometry very accurately**
- **must know 3D -> 2D correspondence**

Method: Setup a linear system

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Solve for m's entries using linear least squares

Ax=0 form

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Just like how you solved for homography!

Can we factorize M back to K [R | T]?

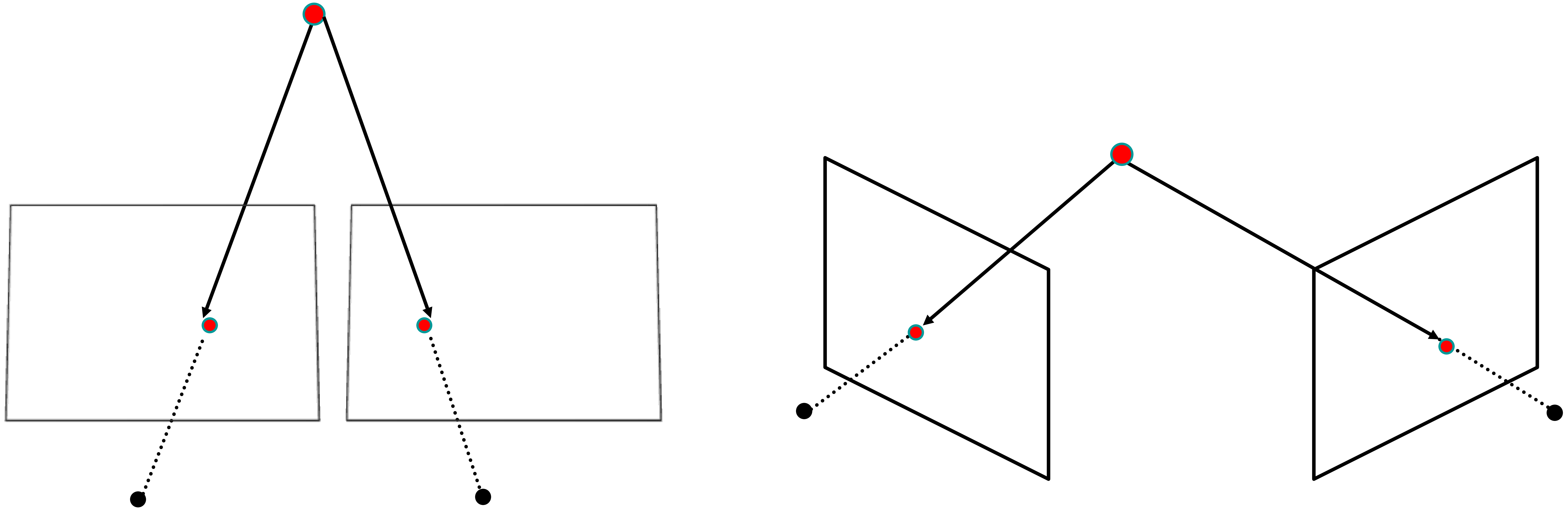
- Yes.
- Why? because K and R have a very special form:

$$\begin{bmatrix} f_x & s & o_x \\ 0 & f_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

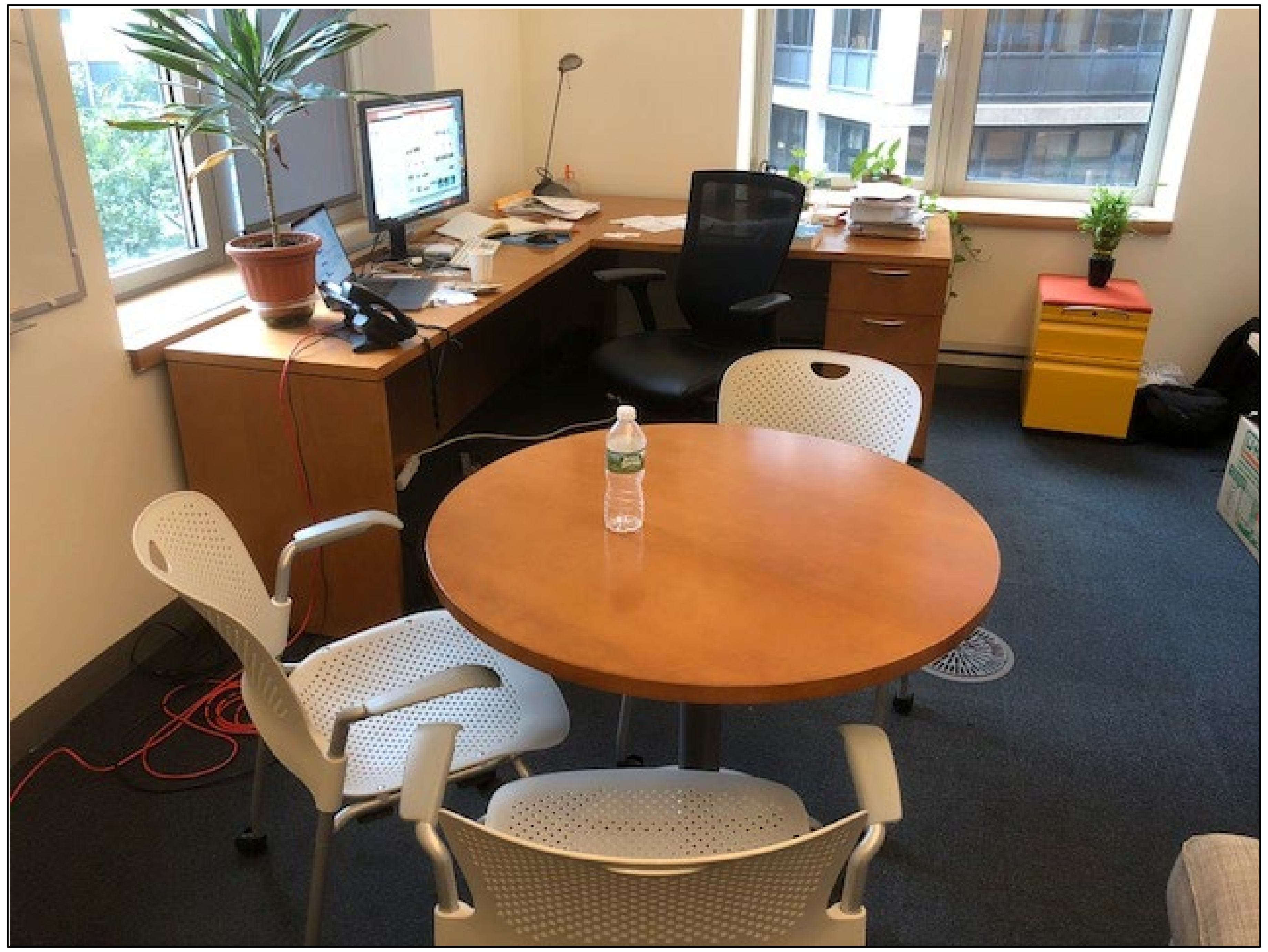
- RQ decomposition
- Practically, use camera calibration packages (there is a good one in OpenCV)

General case

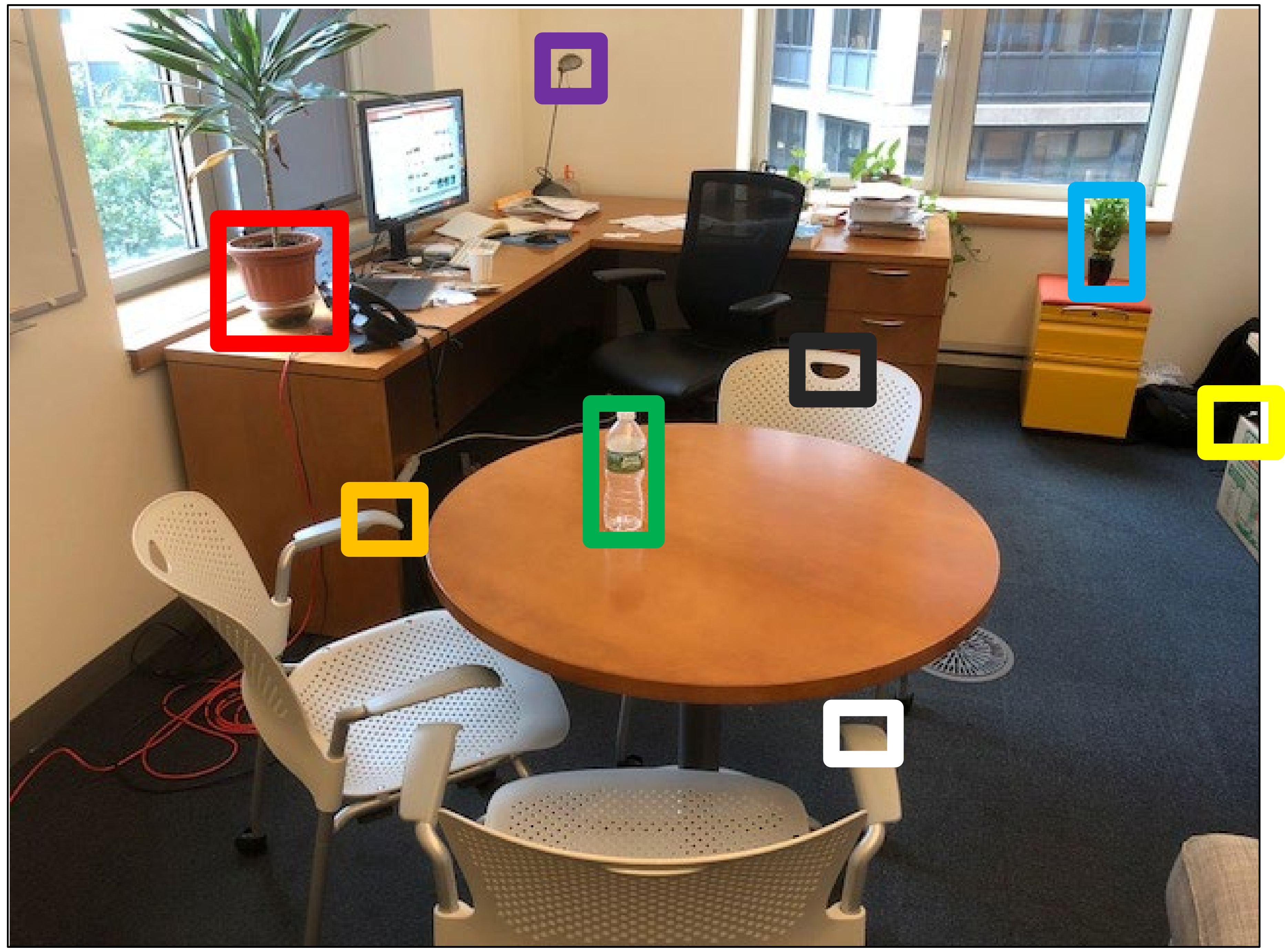
- The two cameras need not have parallel optical axes.



Source: Torralba, Isola, Freeman

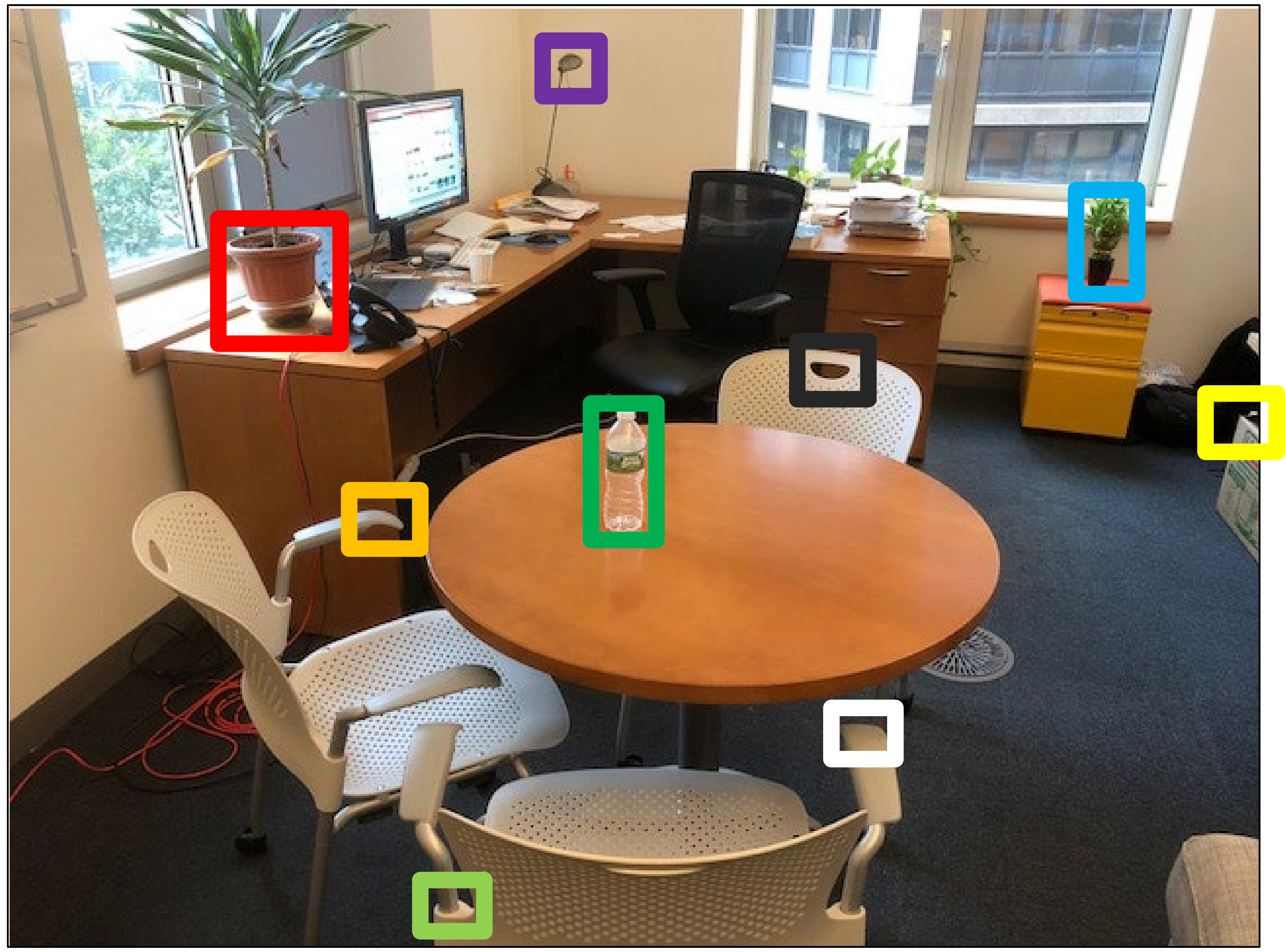


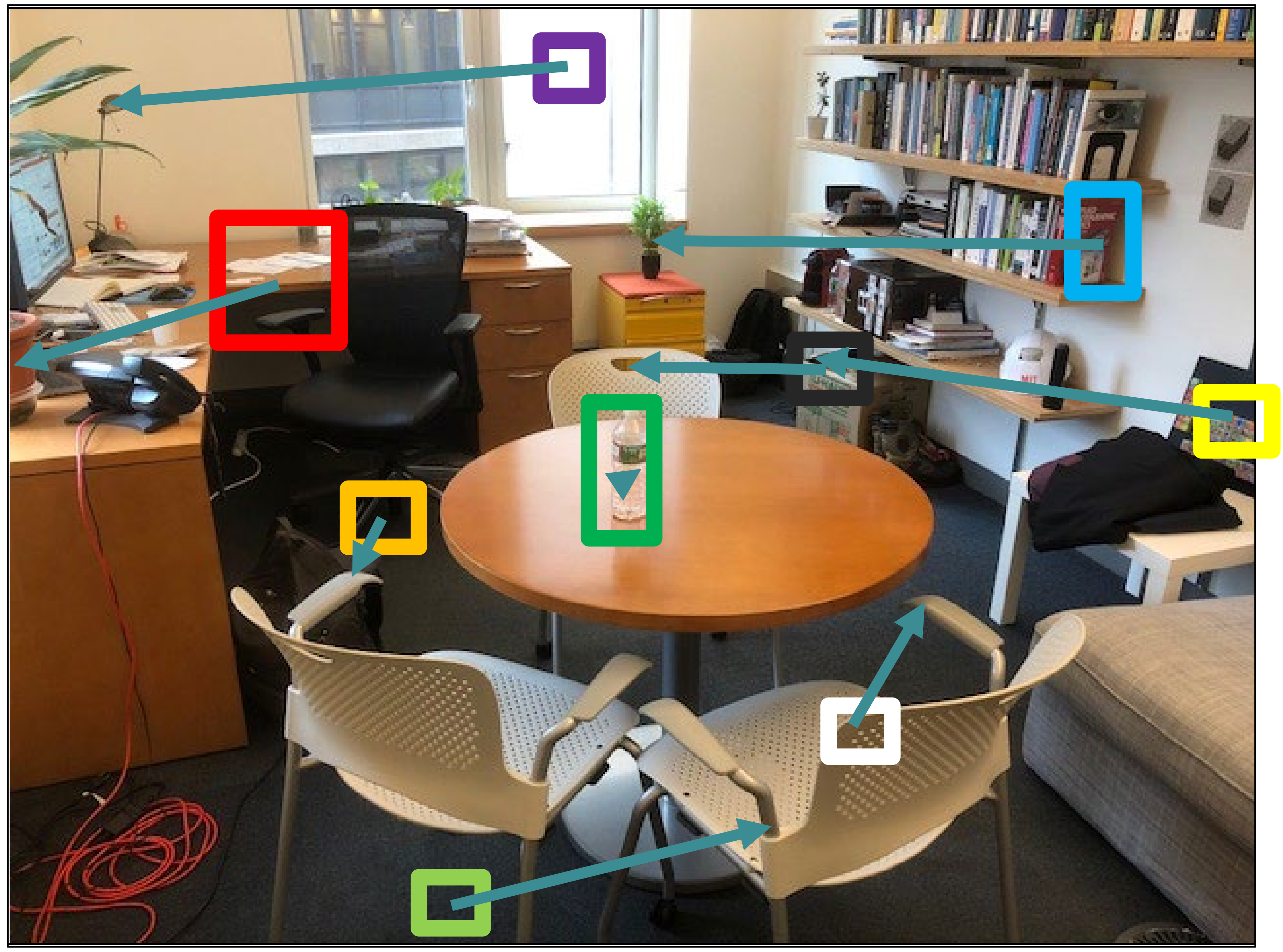
Source: Torralba, Isola, Freeman



Can we search for matches only along horizontal lines?

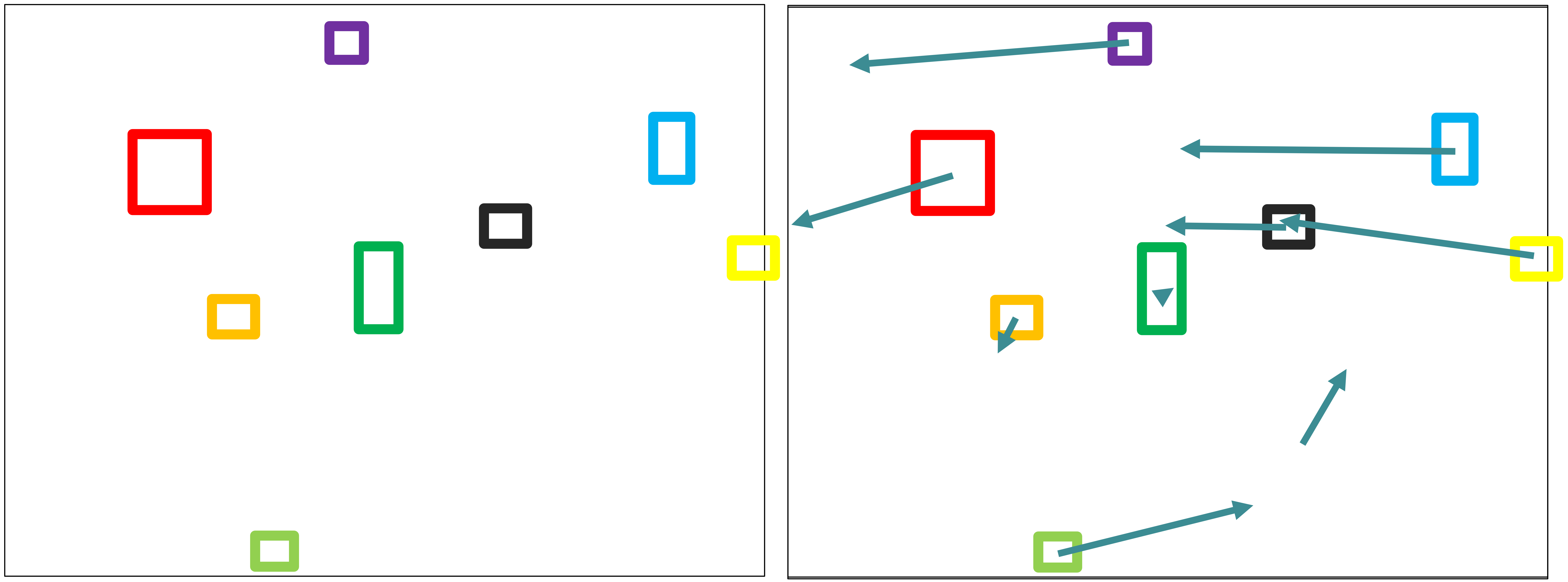
Source: Torralba, Isola, Freeman





Can we search for matches only along horizontal lines?

Source: Torralba, Isola, Freeman

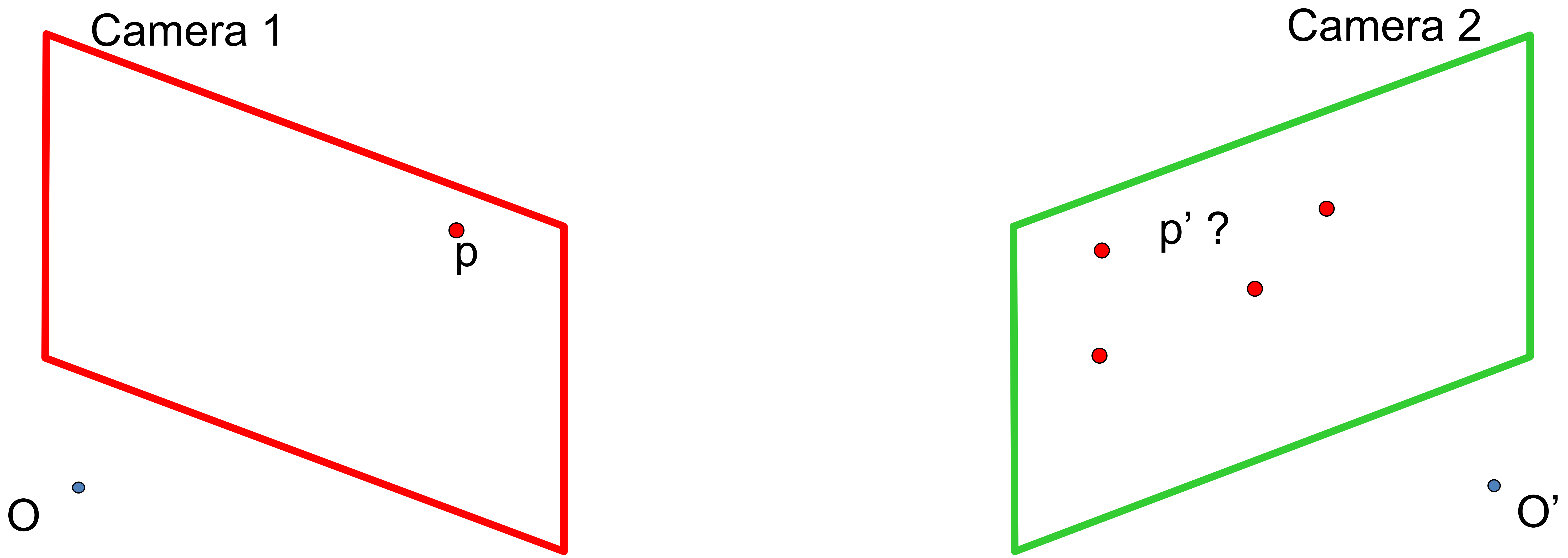


Can we search for matches only along horizontal lines?

It looks like we might need to search everywhere... are there any constraints that can guide the search?

Source: Torralba, Isola, Freeman

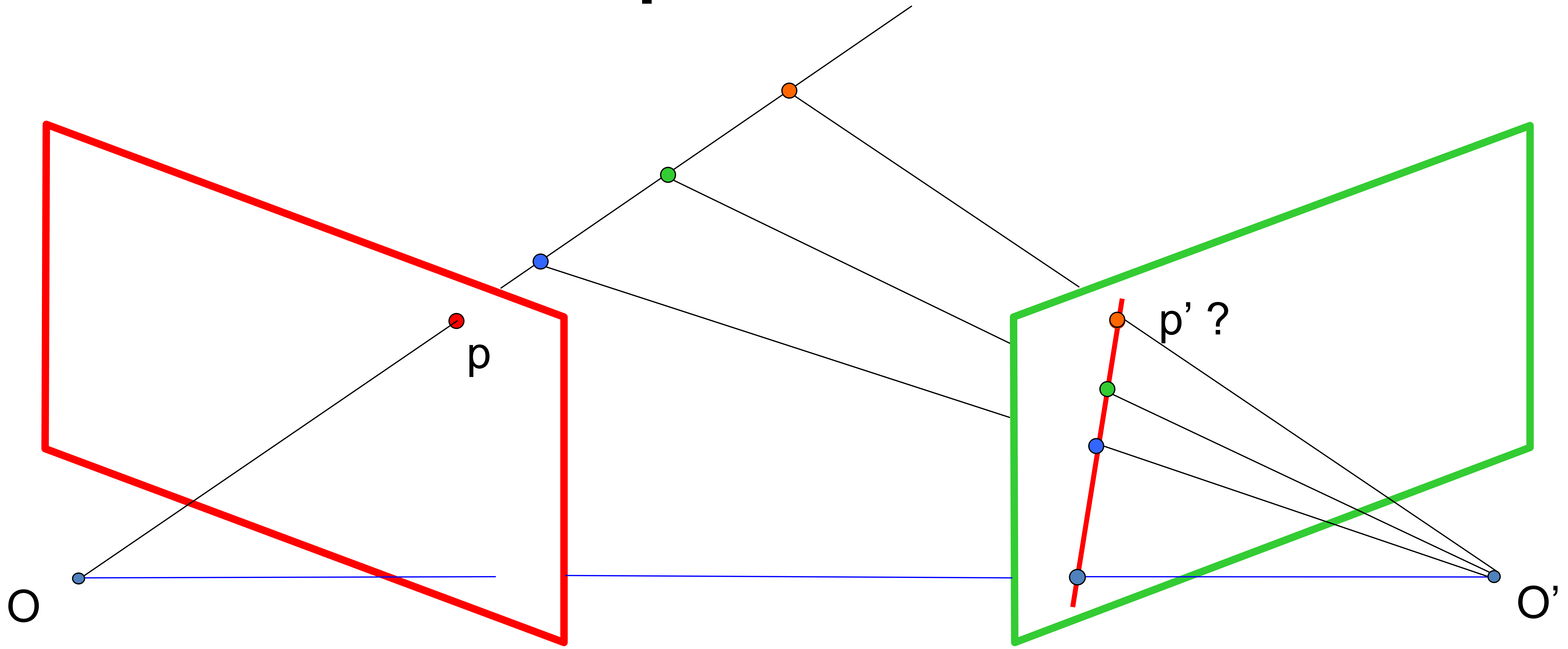
Stereo correspondence constraints



If we see a point in camera 1, are there any constraints on where we will find it on camera 2?

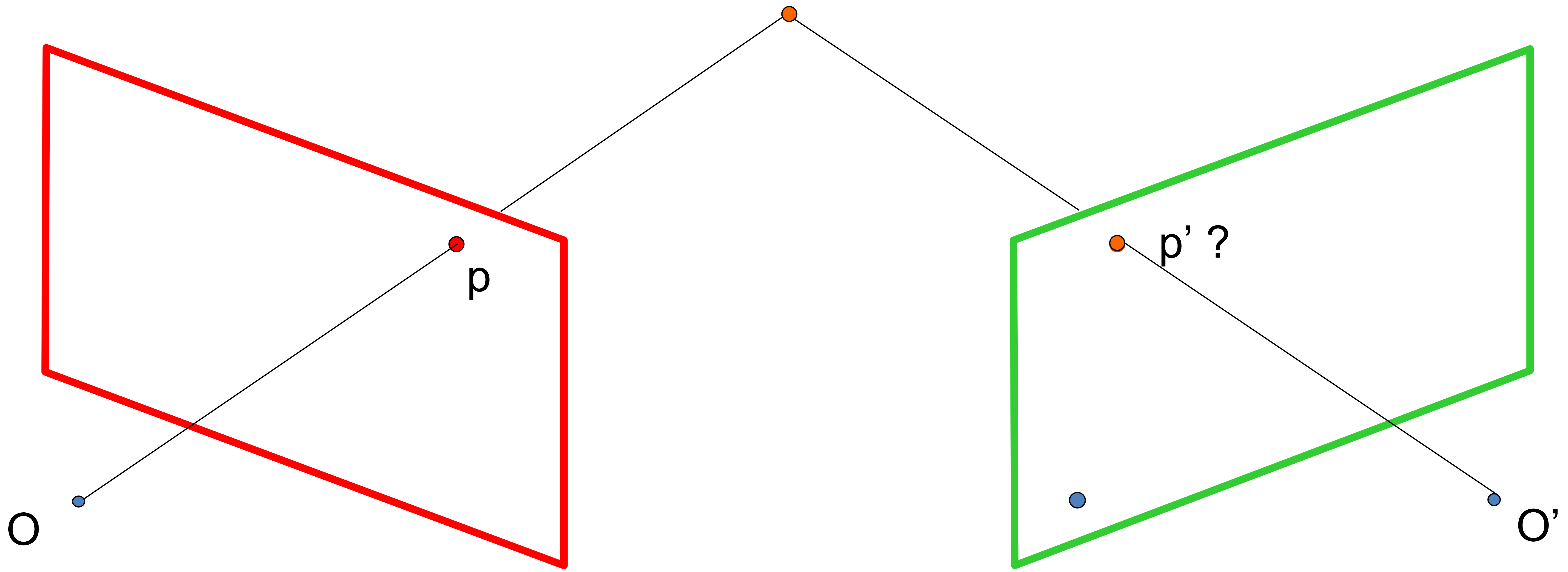
Source: Torralba, Isola, Freeman

Stereo correspondence constraints



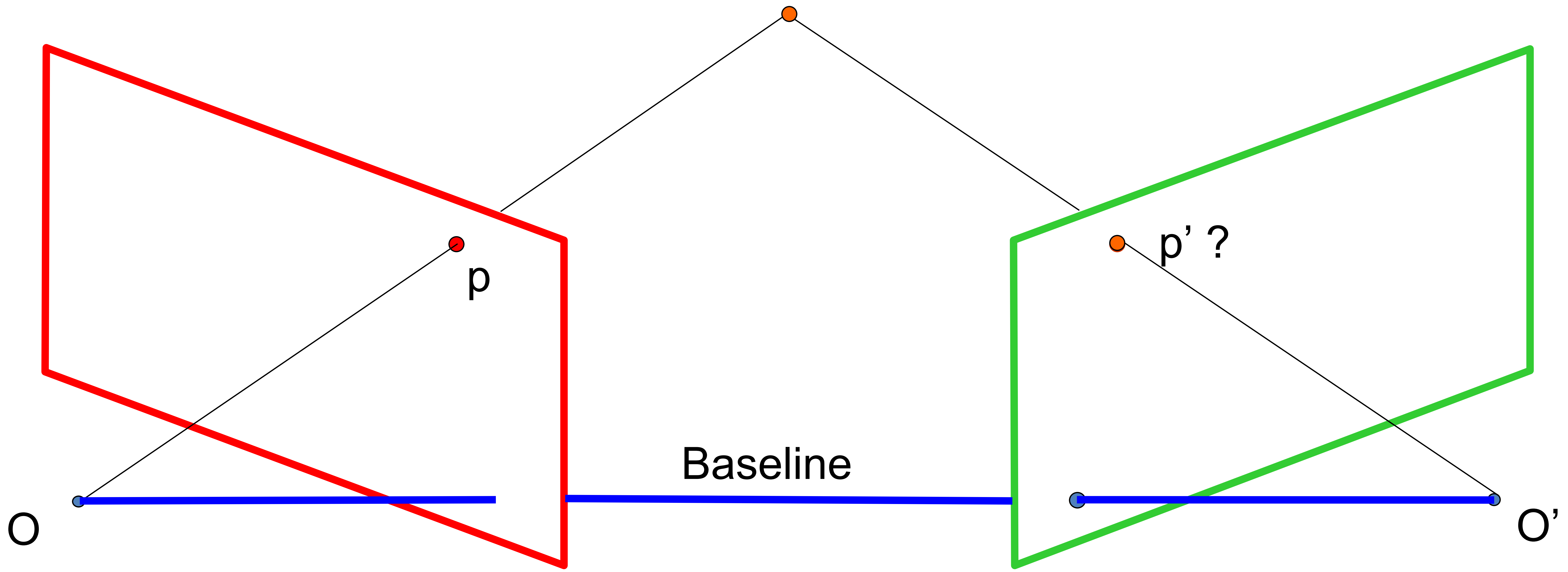
Source: Torralba, Isola, Freeman

Some terminology



Source: Torralba, Isola, Freeman

Some terminology

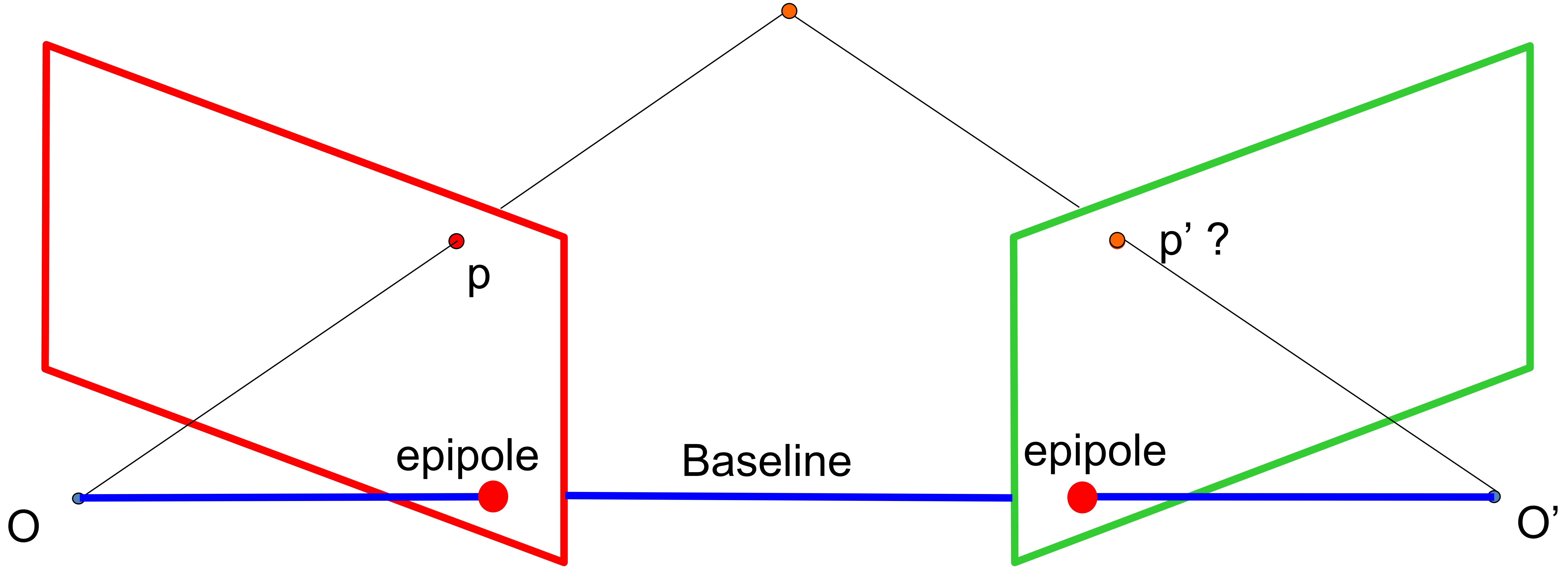


Baseline: the line connecting the two camera centers

Epipole: point of intersection of *baseline* with the image plane

Source: Torralba, Isola, Freeman

Some terminology

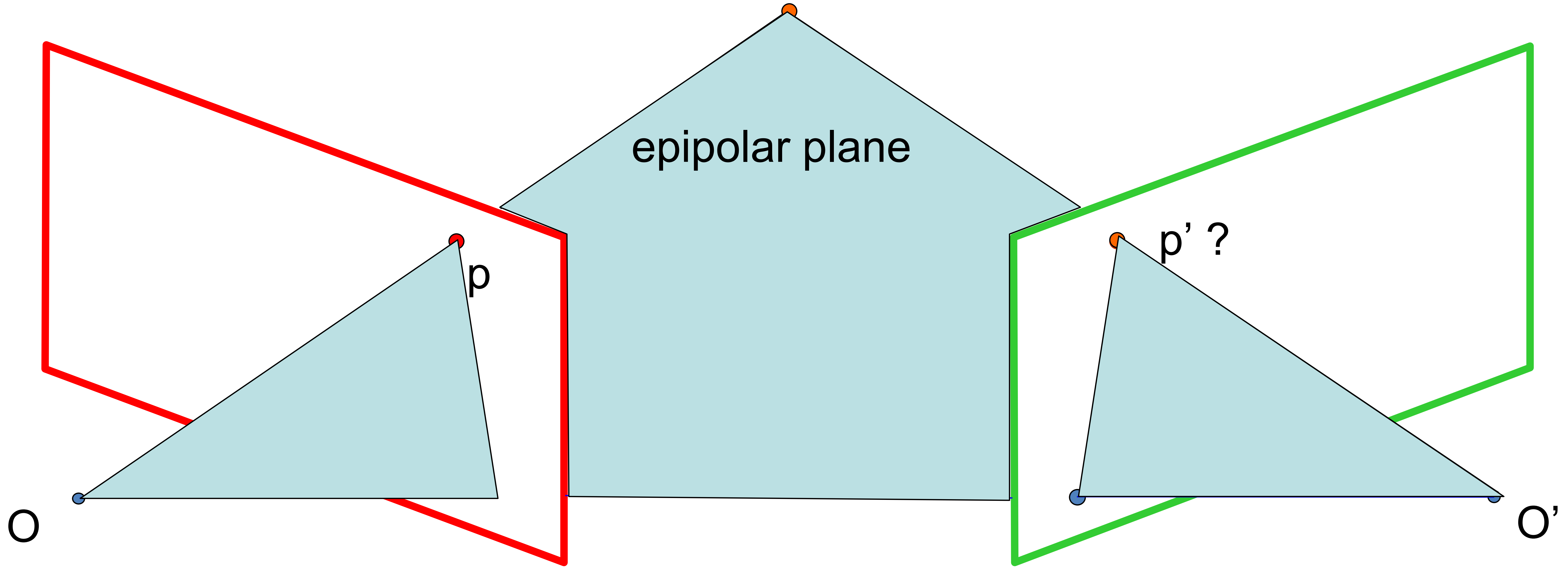


Baseline: the line connecting the two camera centers

Epipole: point of intersection of *baseline* with the image plane

Source: Torralba, Isola, Freeman

Some terminology



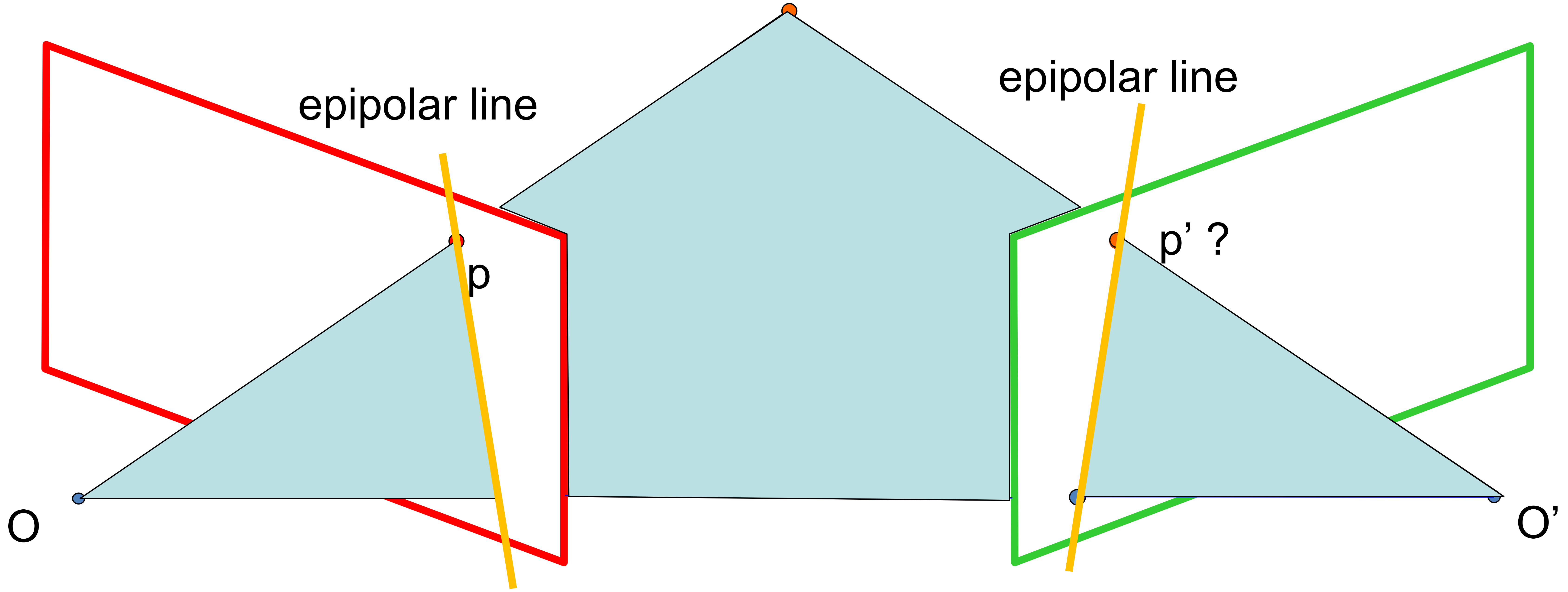
Baseline: the line connecting the two camera centers

Epipole: point of intersection of *baseline* with the image plane

Epipolar plane: the plane that contains the two camera centers and a 3D point in the world

Source: Torralba, Isola, Freeman

Some terminology



Baseline: the line connecting the two camera centers

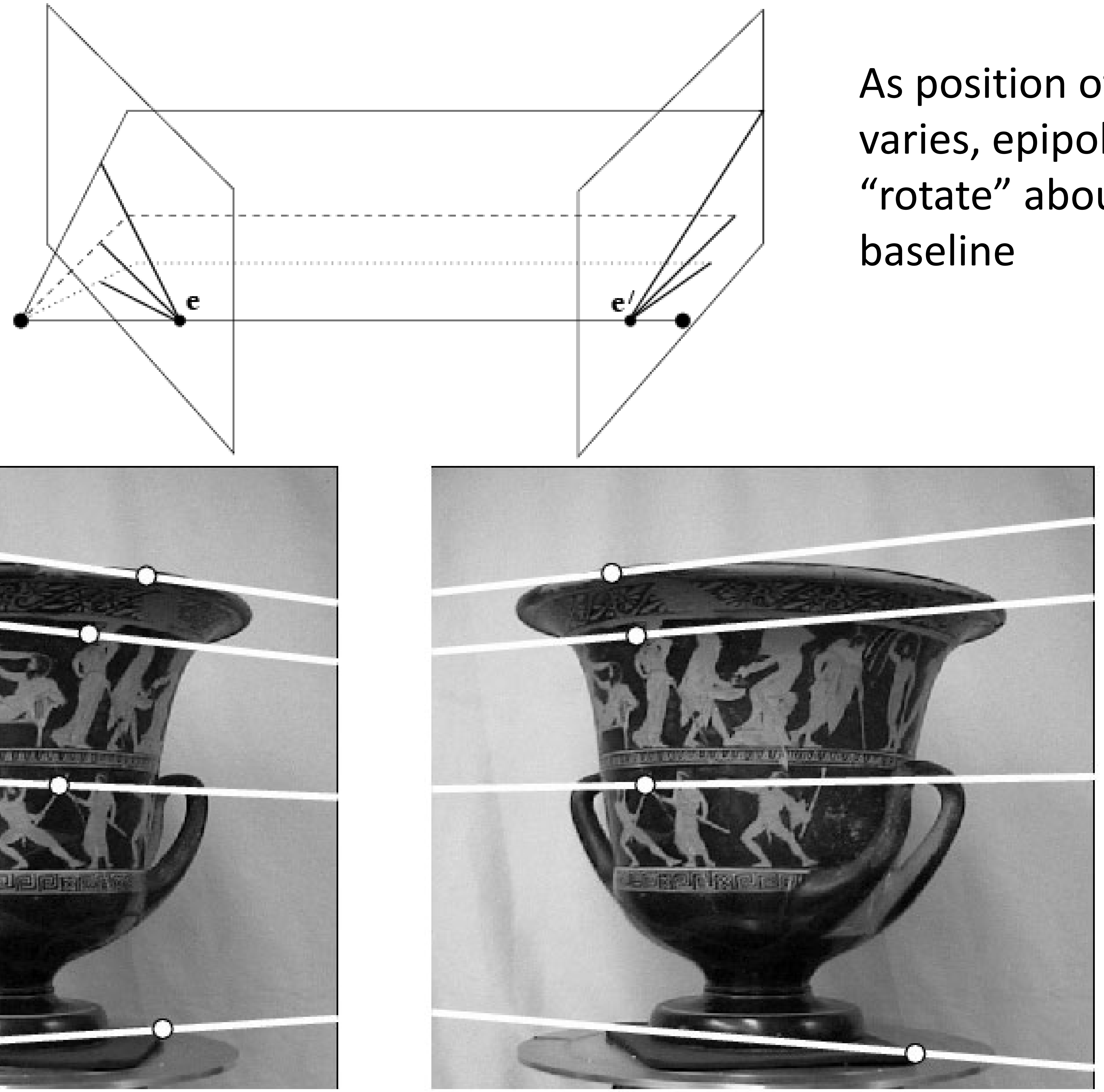
Epipole: point of intersection of *baseline* with the image plane

Epipolar plane: the plane that contains the two camera centers and a 3D point in the world

Epipolar line: intersection of the *epipolar plane* with each image plane

Source: Torralba, Isola, Freeman

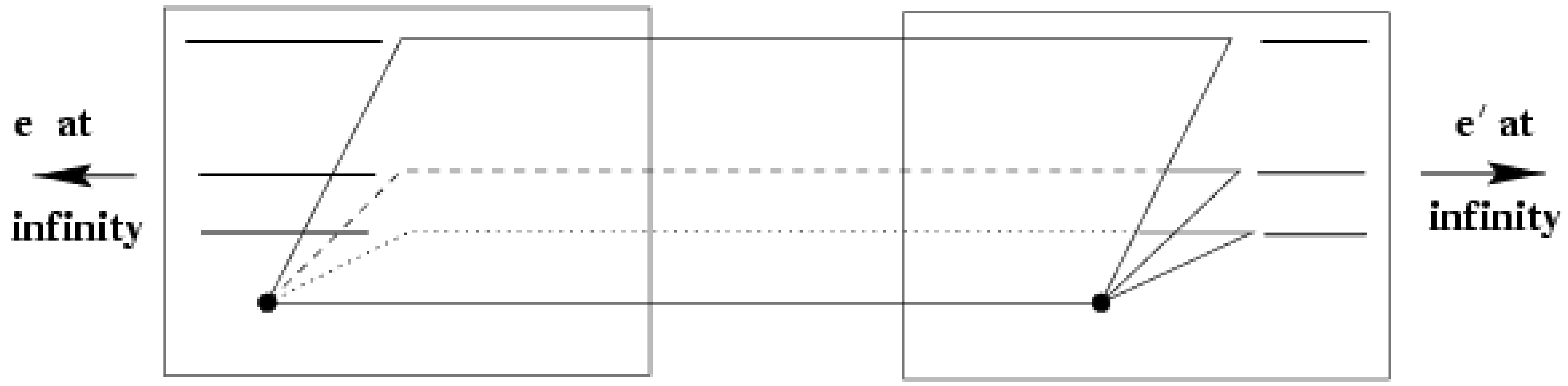
Example: converging cameras



As position of 3d point varies, epipolar lines “rotate” about the baseline

Figure from Hartley & Zisserman

Example: motion parallel with image plane



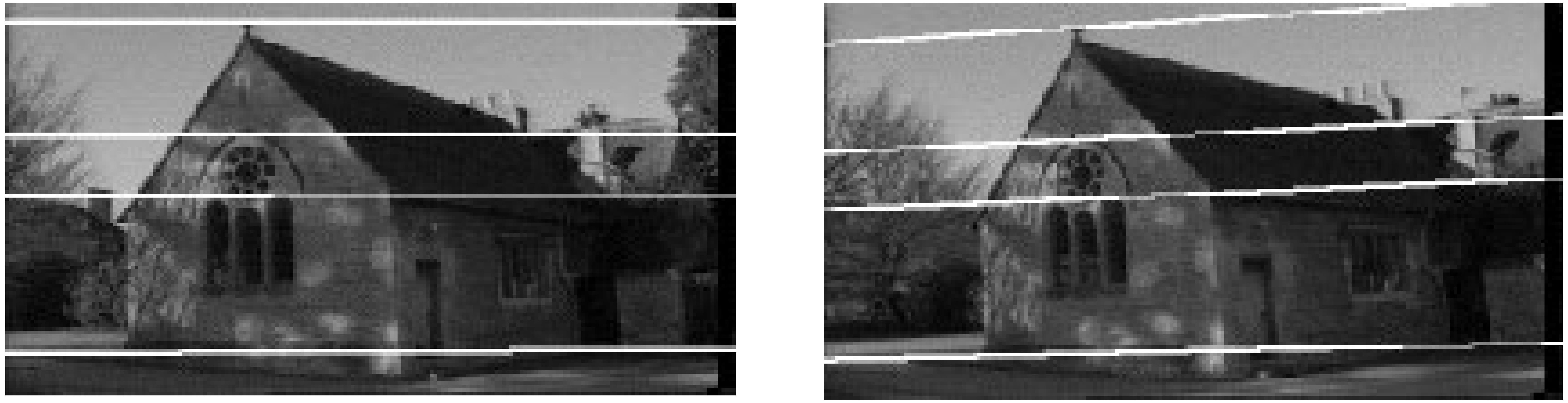
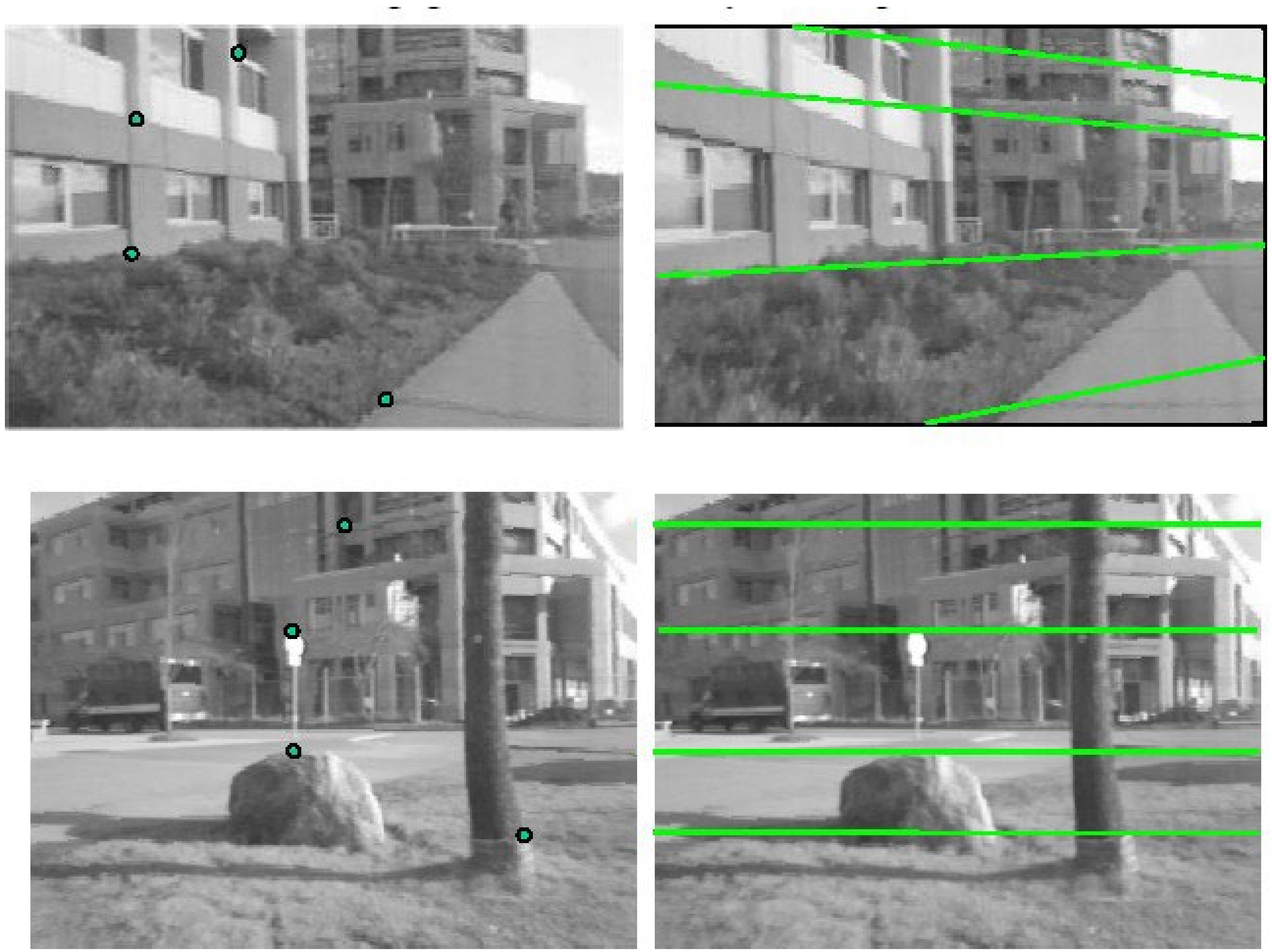
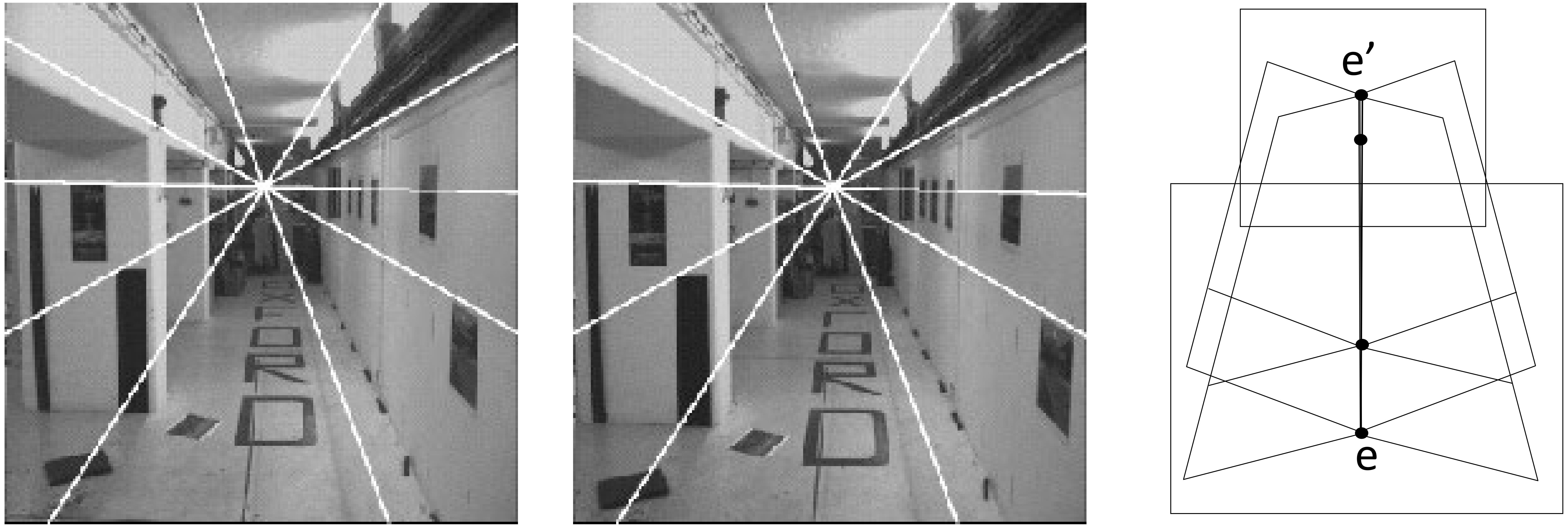


Figure from Hartley & Zisserman

Example



Example: forward motion



Epipole has same coordinates in both images.
Points move along lines radiating from e: “Focus of expansion”

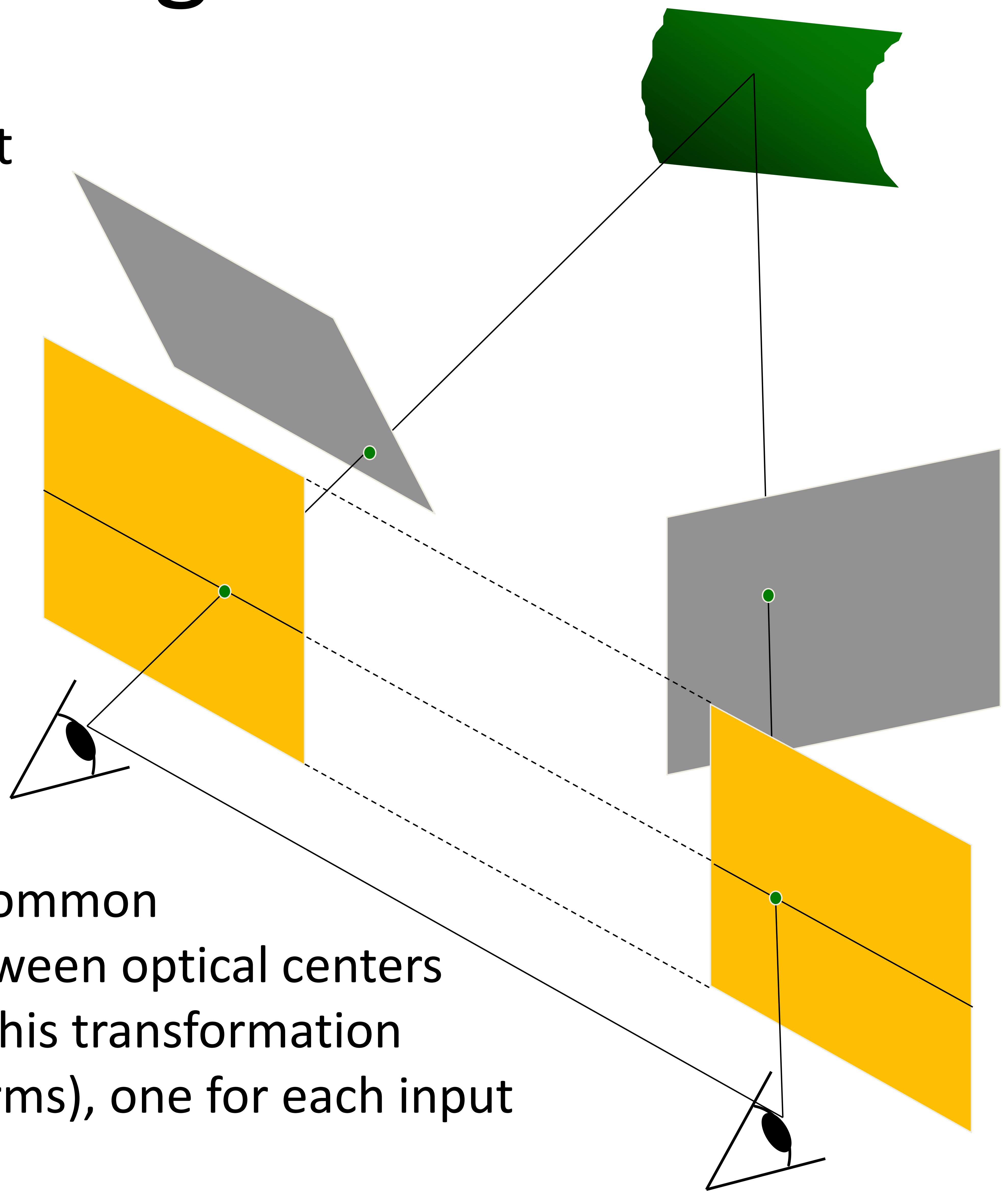
Figure from Hartley & Zisserman

The Epipole

Photo by Frank Dellaert

Stereo image rectification

In practice, it is convenient if image scanlines are the epipolar lines.

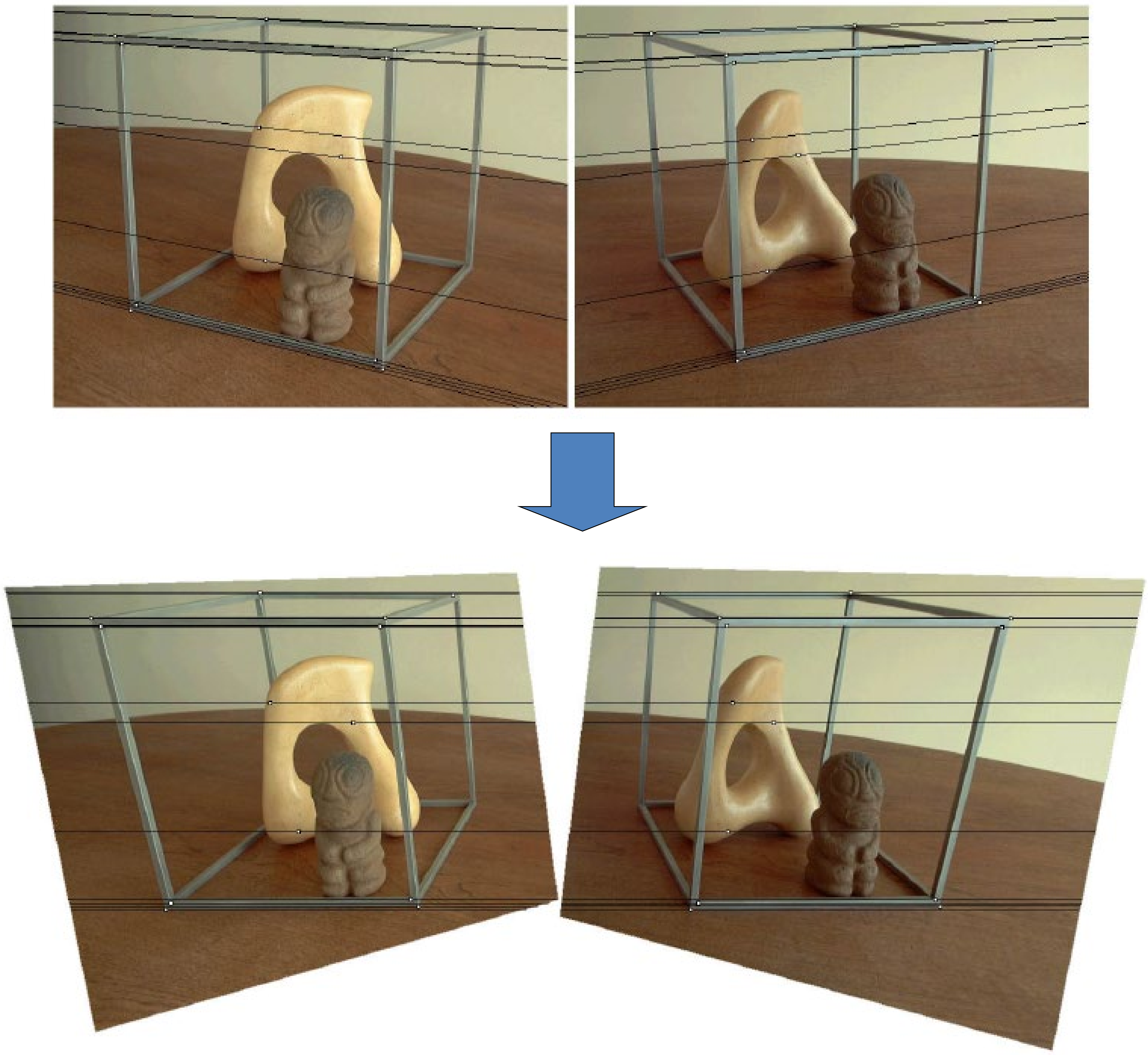


- reproject image planes onto a common plane parallel to the line between optical centers
- pixel motion is horizontal after this transformation
- two homographies (3x3 transforms), one for each input image reprojection

Adapted from Li Zhang

C. Loop and Z. Zhang. [Computing Rectifying Homographies for Stereo Vision](). CVPR 1999.

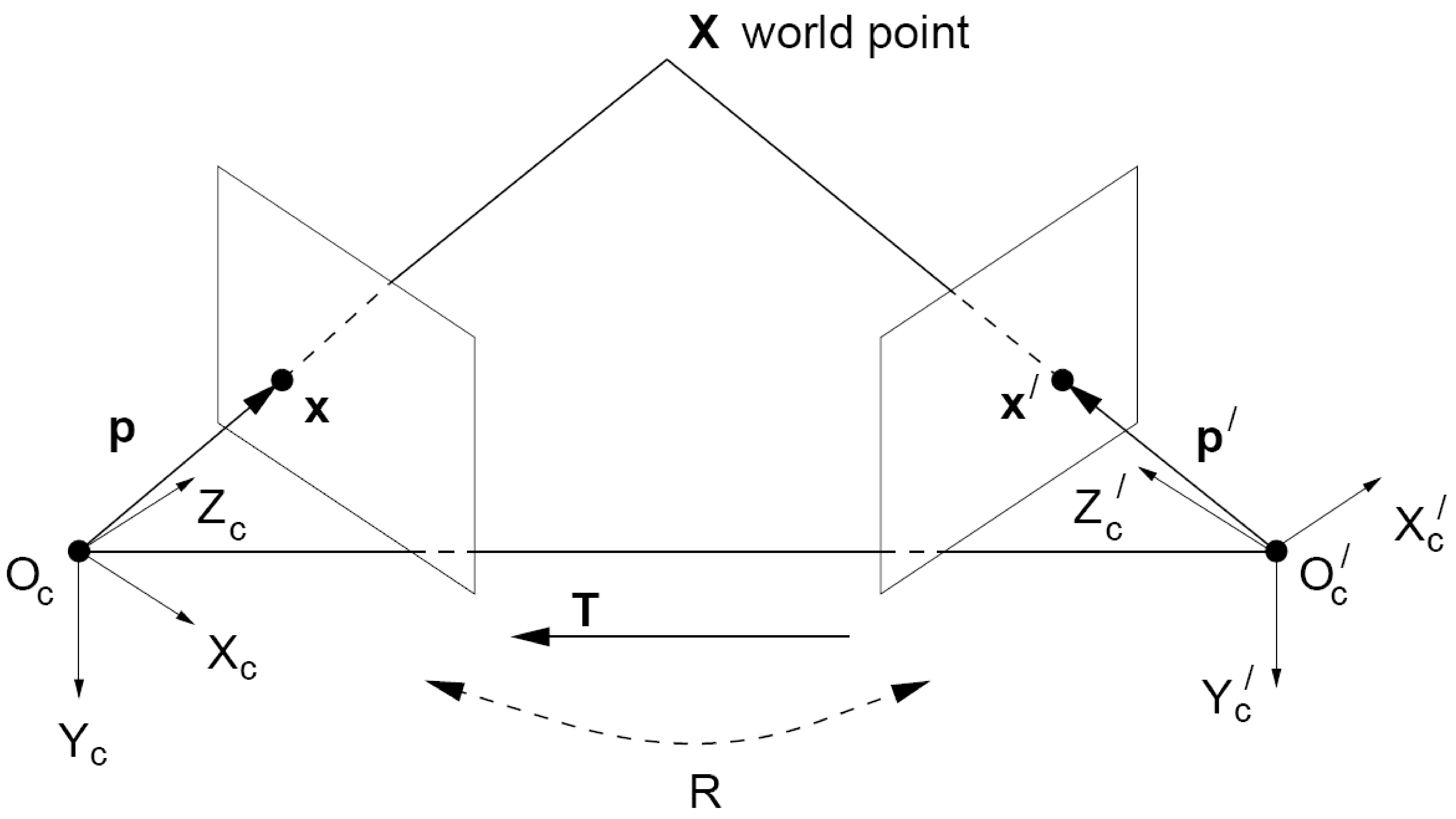
Stereo image rectification: example



- For a given stereo rig, how do we express the epipolar constraints algebraically?

- For calibrated cameras, with **Essential Matrix**
- For uncalibrated cameras, with **Fundamental Matrix**

Deriving the Essential Matrix:
Stereo geometry, with calibrated cameras



If the rig is calibrated, we know :

how to **rotate** and **translate** camera reference frame 1 to get to camera reference frame 2.

Rotation: 3 x 3 matrix; translation: 3 vector.

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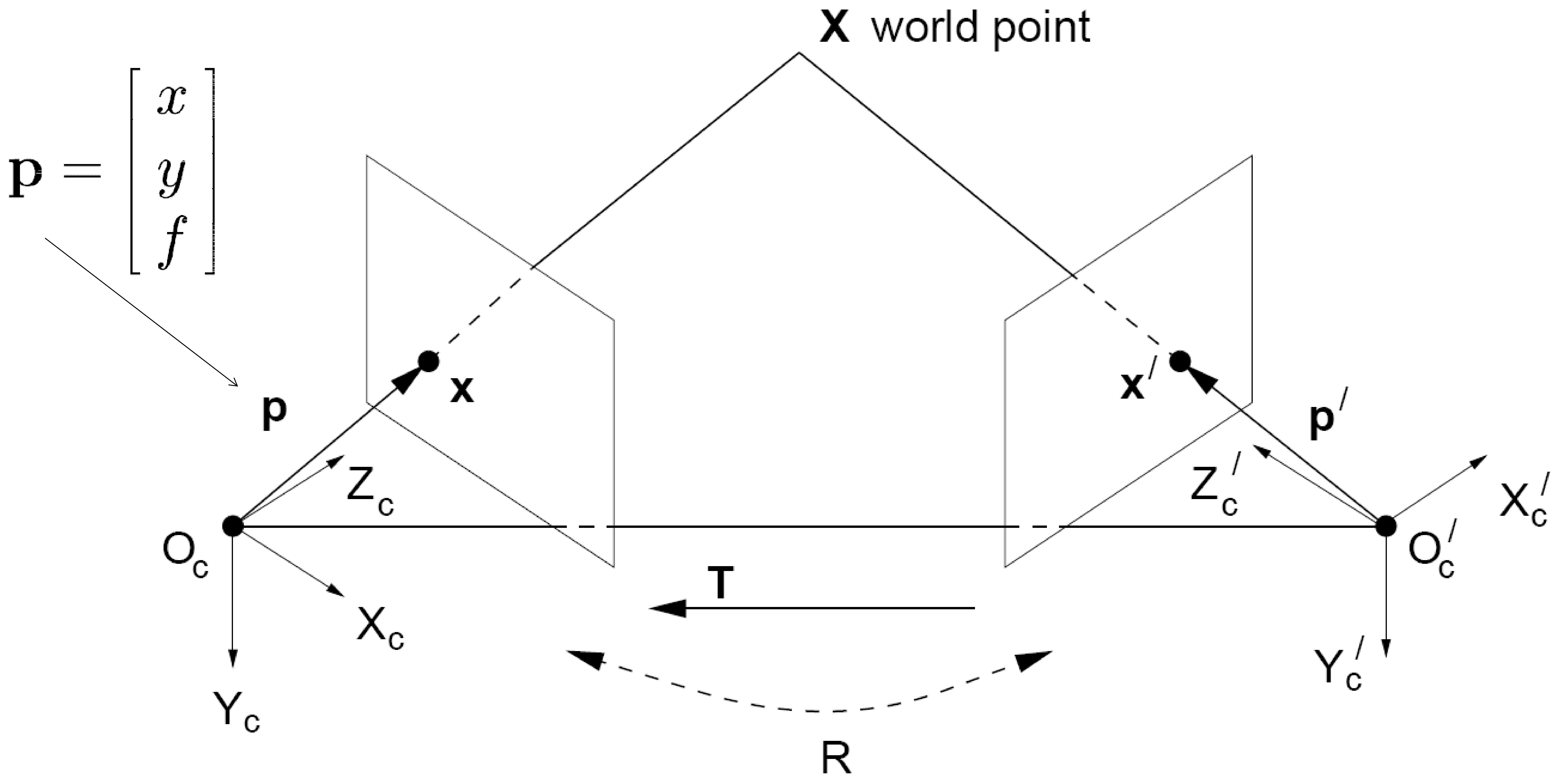
3d rigid transformation

$$\mathbf{X}' = \mathbf{R}\mathbf{X} + \mathbf{T}$$

Deriving the Essential Matrix:

Stereo geometry, with calibrated cameras



Camera-centered coordinate systems are related by known rotation **R** and translation **T**:

$$\mathbf{X}' = \mathbf{R}\mathbf{X} + \mathbf{T}$$

Grauman

Review: Cross product

$$\vec{a} \times \vec{b} = \vec{c}$$

$$\vec{a} \cdot \vec{c} = 0$$

$$\vec{b} \cdot \vec{c} = 0$$

Vector cross product takes two vectors and returns a third vector that’s perpendicular to both inputs.

So here, c is perpendicular to both a and b, which means the dot product = 0.

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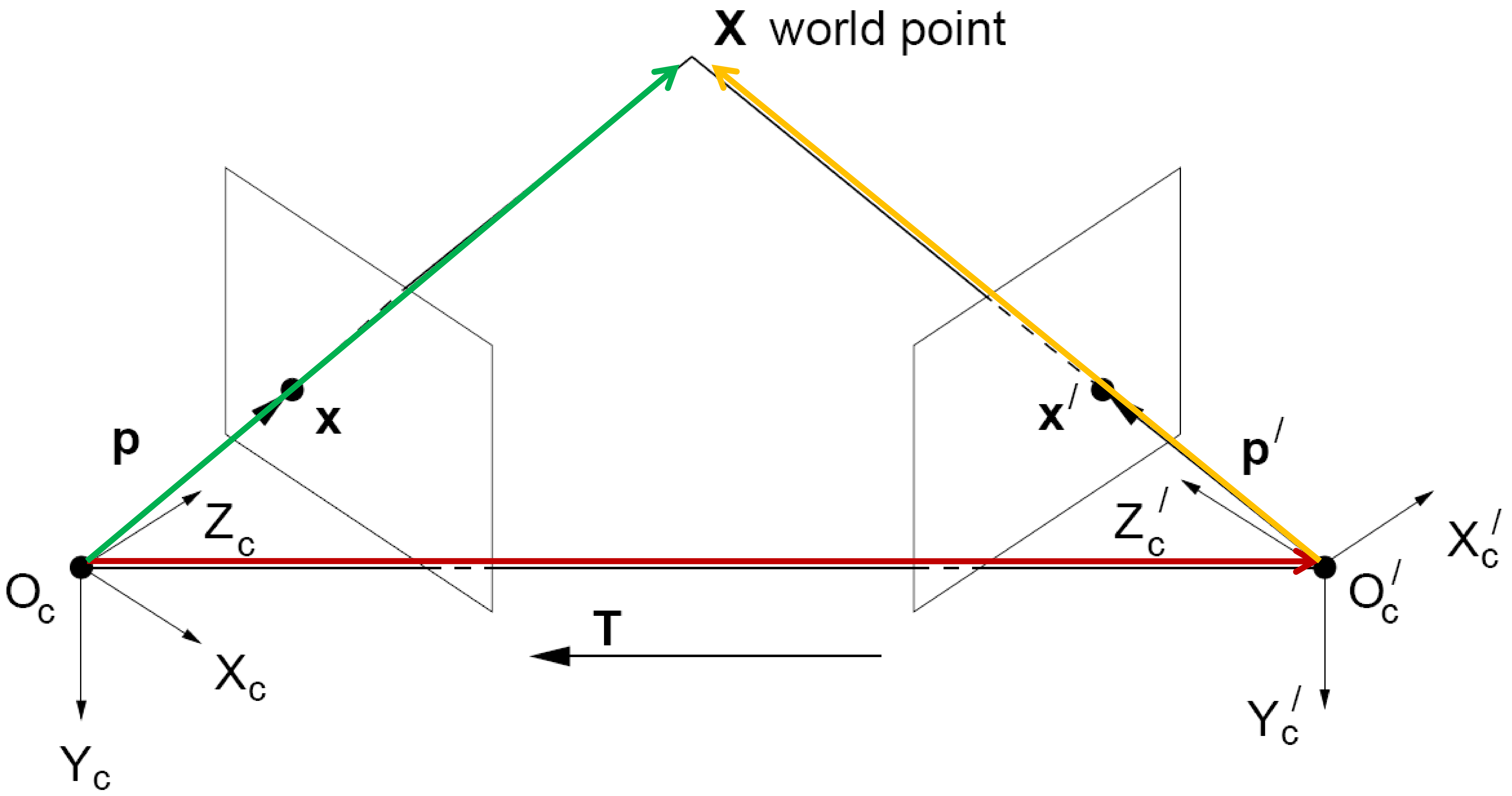
Deriving the Essential Matrix:

From geometry to algebra



$$\mathbf{X}' = \mathbf{R}\mathbf{X} + \mathbf{T}$$

$$\mathbf{T} \times \mathbf{X}' =$$

Normal to the plane

$$= \mathbf{T} \times \mathbf{R}\mathbf{X}$$

$$\mathbf{X}' \cdot (\mathbf{T} \times \mathbf{X}') = \mathbf{X}' \cdot (\mathbf{T} \times \mathbf{R}\mathbf{X})$$

$$= 0$$

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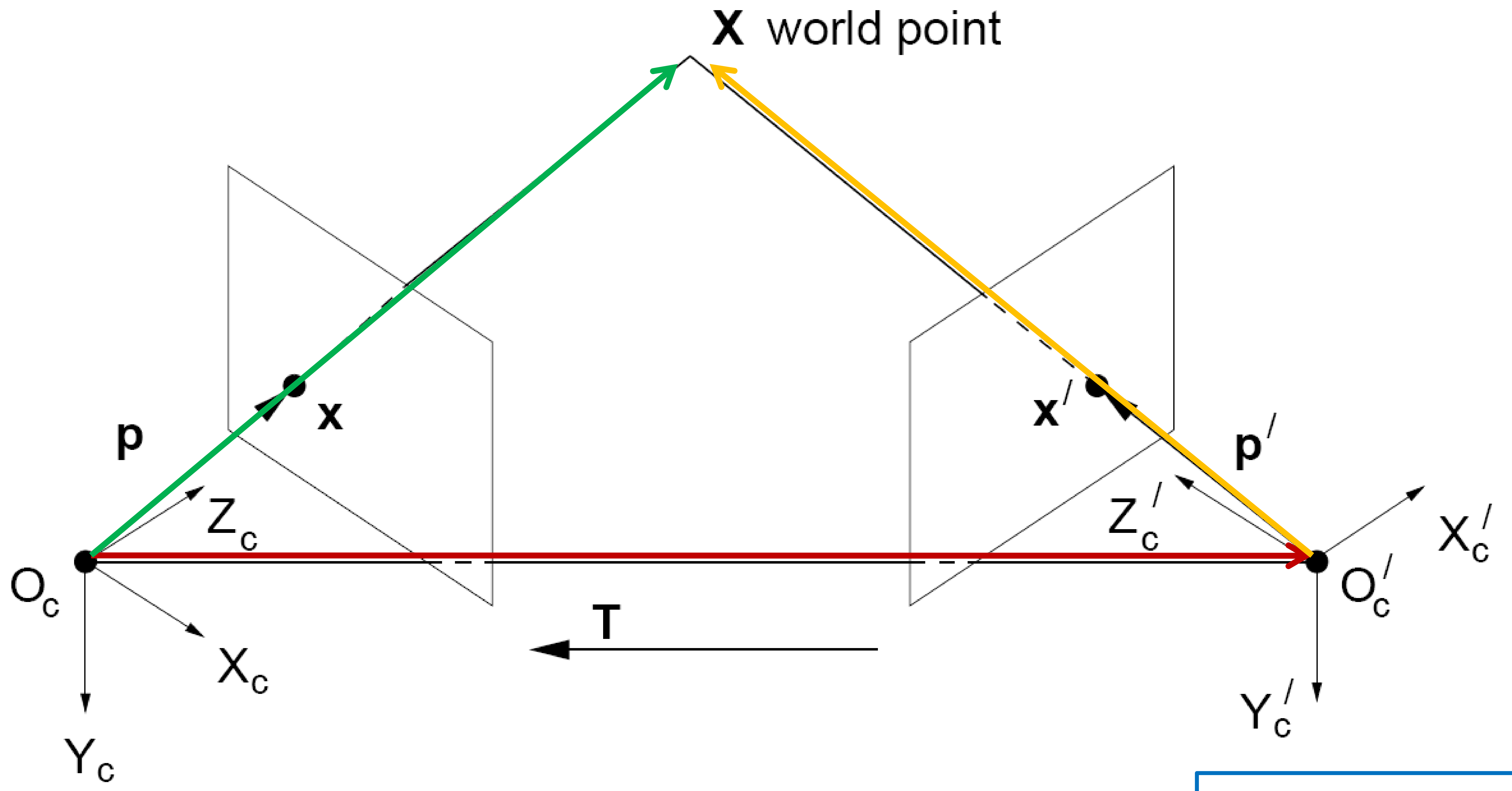
Deriving the Essential Matrix:

From geometry to algebra



$$\mathbf{X'} = \mathbf{R}\mathbf{X} + \mathbf{T}$$

$$\underbrace{\mathbf{T}\times\mathbf{X'}}_{\text{Normal to the plane}} = \mathbf{T}\times\mathbf{R}\mathbf{X} + \mathbf{T}\times\mathbf{T}$$

$$= \mathbf{T}\times\mathbf{R}\mathbf{X}$$

$$\mathbf{X'}\cdot(\mathbf{T}\times\mathbf{X'}) = \mathbf{X'}\cdot(\mathbf{T}\times\mathbf{R}\mathbf{X})$$

$$= 0$$

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Deriving the Essential Matrix:

Matrix form of cross product

$$\vec{a} \times \vec{b} \qquad = \vec{c}$$

$$\vec{a} \cdot \vec{c} = 0$$

$$\vec{b} \cdot \vec{c} = 0$$

Can be expressed as a matrix multiplication.

$$[a_x] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\vec{a} \times \vec{b} = [a_x]\vec{b}$$

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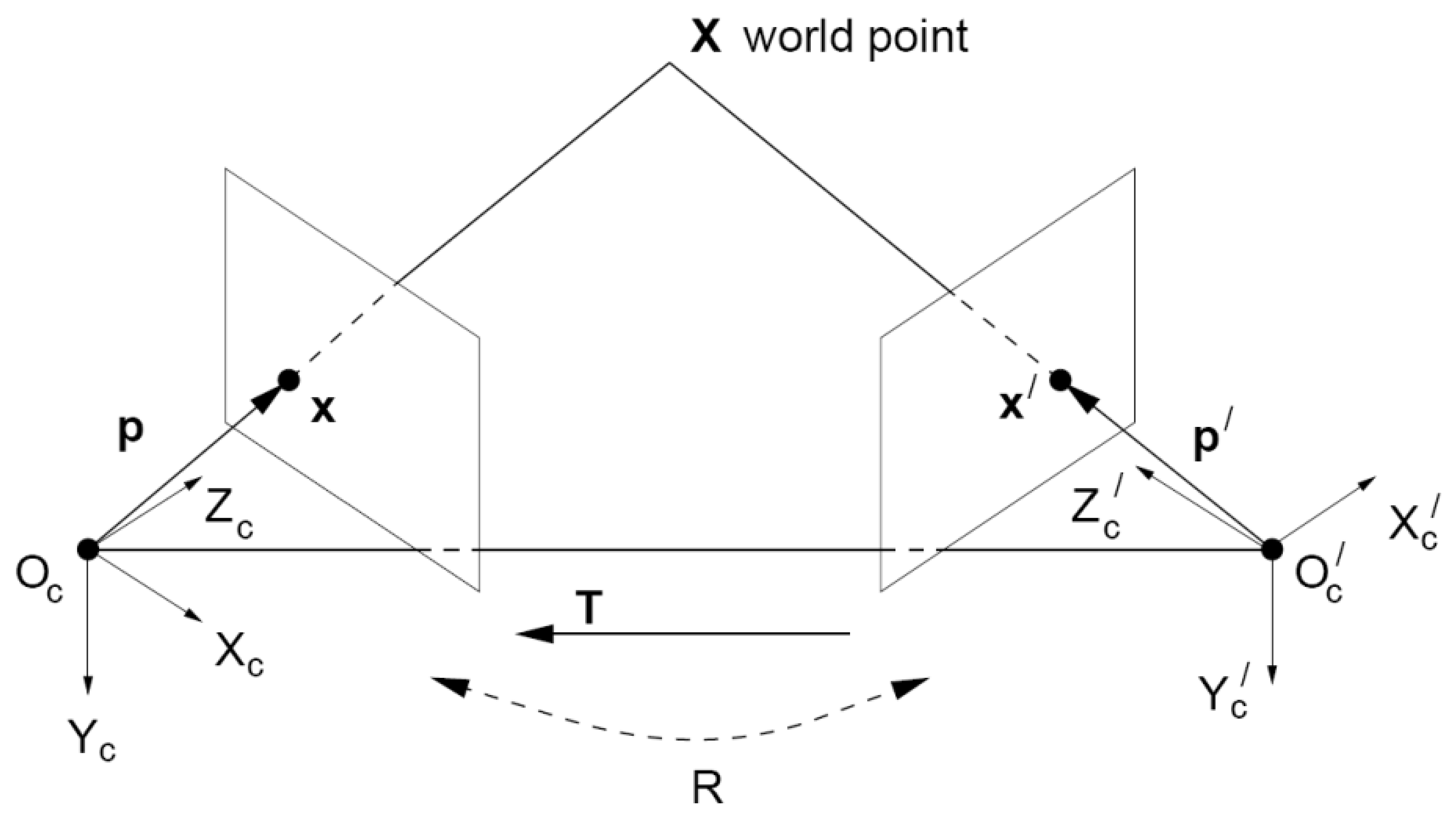
Essential matrix

$$\mathbf{X}' \cdot (\mathbf{T}_x \ \mathbf{R}\mathbf{X}) = 0$$

Let $\mathbf{E} = \mathbf{T}_x\mathbf{R}$



This holds for the rays **p** and **p’** that are parallel to the camera-centered position vectors **X** and **X’**, so we have:

$$\mathbf{p}'^T \mathbf{E}\mathbf{p} = 0$$

E is called the **essential matrix**, which relates corresponding image points [Longuet-Higgins 1981]

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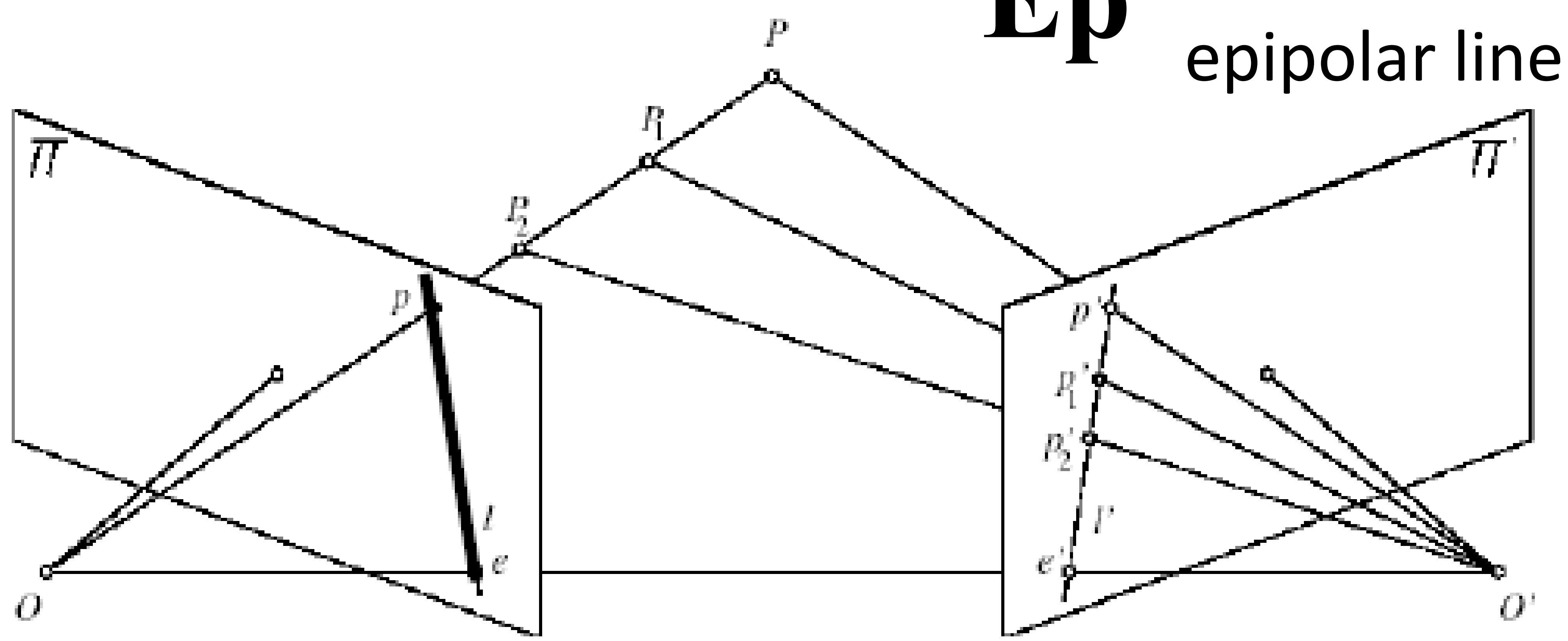
Essential matrix and epipolar lines

$$\mathbf{p}'^{\mathrm{T}}\mathbf{E}\mathbf{p} = 0$$

Epipolar constraint: if we observe point **p** in one image, then its position **p’** in second image must satisfy this equation.

$\mathbf{E}\mathbf{p}$ is the coordinate vector representing the epipolar line associated with point p



$\mathbf{E}^T\mathbf{p}'$ is the coordinate vector representing the epipolar line associated with point p’

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Essential matrix: properties

- Relates image of corresponding points in both cameras, given rotation and translation
- Assuming intrinsic parameters are known

$$\mathbf{E} = \mathbf{T}_x \mathbf{R}$$

- $E\,x'$ is the epipolar line associated with x' ($l = E\,x'$)
- $E^T x$ is the epipolar line associated with x ($l' = E^T x$)
- $E\,e' = 0$ and $E^T e = 0$
- E is singular (rank two)
- E has five degrees of freedom
  - (3 for R, 2 for t because it's up to a scale)

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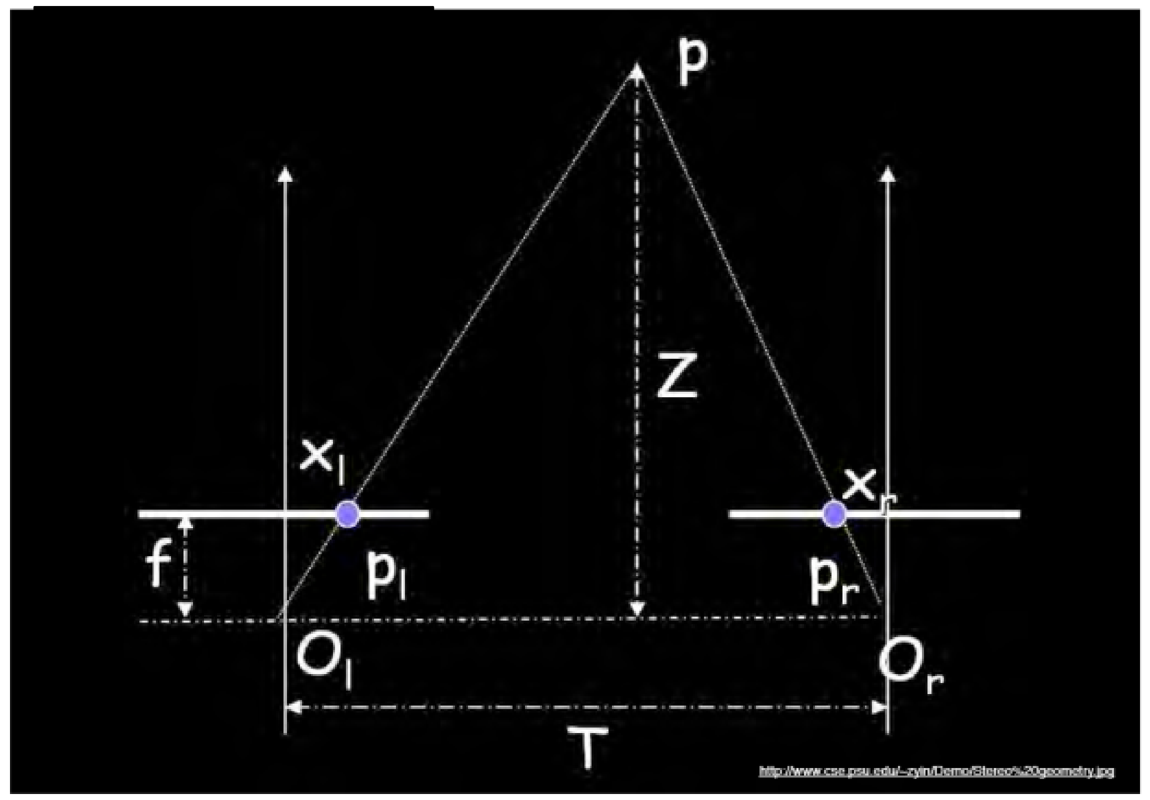
Essential matrix example: parallel cameras



$$\mathbf{R} =$$

$$\mathbf{T} =$$

$$\mathbf{E} = [\mathbf{T}_{\mathbf{x}}]\mathbf{R} =$$

$$\mathbf{p}'^{\mathrm{T}}\mathbf{E}\mathbf{p} = 0$$

For the parallel cameras, image of any point must lie on same horizontal line in each image plane.

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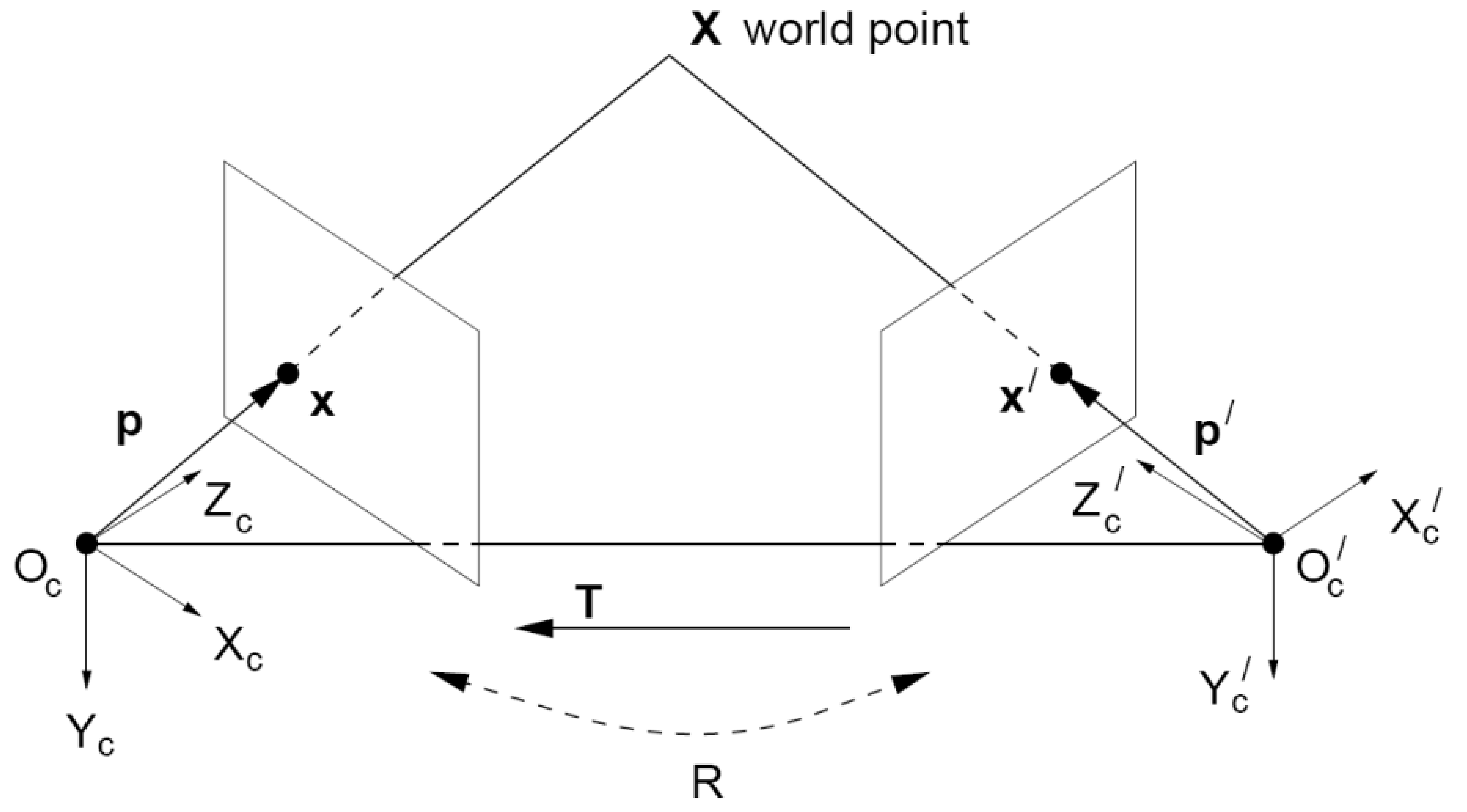
Weak calibration

- So far, we have assumed calibrated cameras and were able to perform dense stereo estimation
- What if we want to estimate world geometry without requiring calibrated cameras?
  - Archival videos
  - Photos from multiple unrelated users
  - Dynamic camera system

Uncalibrated cameras

$$\mathbf{E} = \mathbf{T}_x\mathbf{R}$$

$$\mathbf{p'}^T \mathbf{E}\mathbf{p} = 0$$



- For an *uncalibrated* stereo rig, can we we express the epipolar constraints algebraically via the **Essential Matrix**?
- No, we do not know T or R
- However we can use the **Fundamental Matrix**
  - Estimate epipolar geometry from a (redundant) set of point correspondences between two uncalibrated cameras

Uncalibrated case

For a given camera:

$$\bar{\mathbf{p}} = \mathbf{M}_{\text{int}}\mathbf{p}$$

← Camera coordinates

So, for two cameras (left and right):

$$\mathbf{p}_{(left)} = \mathbf{M}^{-1}_{left,\text{int}}\bar{\mathbf{p}}_{(left)}$$

$$\mathbf{p}_{(right)} = \mathbf{M}^{-1}_{right,\text{int}}\bar{\mathbf{p}}_{(right)}$$

Camera coordinates

Image pixel coordinates

Internal calibration matrices, one per camera

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Uncalibrated case: Fundamental matrix

$$\mathbf{p}_{(left)} = \mathbf{M}_{left,\text{int}}^{-1} \overline{\mathbf{p}}_{(left)}$$

$$\mathbf{p}_{(right)} = \mathbf{M}_{right,\text{int}}^{-1} \overline{\mathbf{p}}_{(right)}$$

$$^{c}\mathbf{p}_{(right)}^{\mathrm{T}} \mathbf{E} \mathbf{p}_{(left)} = 0$$

From before, the **essential** matrix **E.**

$$\left(\mathbf{M}_{right,\text{int}}^{-1} \overline{\mathbf{p}}_{right}\right)^{\mathrm{T}} \mathbf{E} \left(\mathbf{M}_{left,\text{int}}^{-1} \overline{\mathbf{p}}_{left}\right) = 0$$

$$\overline{\mathbf{p}}_{right}^{\mathrm{T}} \underbrace{\left(\mathbf{M}_{right,\text{int}}^{-\mathrm{T}} \mathbf{E} \mathbf{M}_{left,\text{int}}^{-1}\right)}_{} \overline{\mathbf{p}}_{left} = 0$$

$$\overline{\mathbf{p}}_{right}^{\mathrm{T}} \mathbf{F} \overline{\mathbf{p}}_{left} = 0$$

Fundamental matrix

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Fundamental matrix

- Relates pixel coordinates in the two views
- More general form than essential matrix: we remove need to know intrinsic parameters
- If we estimate fundamental matrix from correspondences in pixel coordinates, can reconstruct epipolar geometry without intrinsic or extrinsic parameters

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Computing F from correspondences

$$\mathbf{F} = \left(\mathbf{M}_{right,\mathrm{int}}^{-\mathrm{T}} \mathbf{E} \mathbf{M}_{left.\mathrm{int}}^{-1}\right)$$

$$\overline{\mathbf{p}}_{right}^{\mathrm{T}} \mathbf{F} \overline{\mathbf{p}}_{left} = 0$$

- Cameras are uncalibrated: we don’t know **E** or left or right $\mathbf{M}_{\mathrm{int}}$ matrices
- Estimate F from 8+ point correspondences.

Computing F from correspondences

Each point correspondence generates one constraint on F

$$\overline{\mathbf{p}}_{right}^{\mathrm{T}} \mathbf{F} \overline{\mathbf{p}}_{left} = 0$$

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

Collect n of these constraints

$$\begin{bmatrix} u'_1 u_1 & u'_1 v_1 & u'_1 & v'_1 u_1 & v'_1 v_1 & v'_1 & u_1 & v_1 & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = \mathbf{0}$$

Solve for f , vector of parameters.

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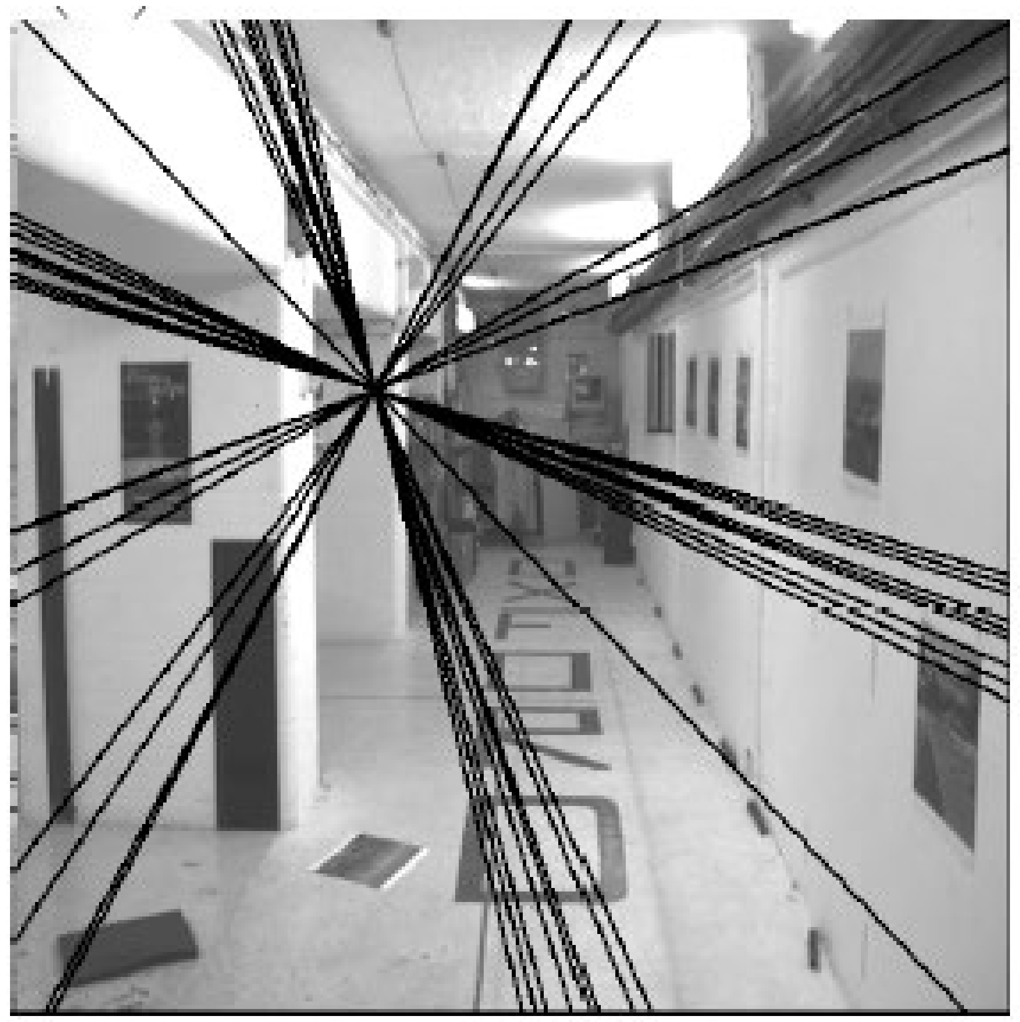
Rank constraint

$$\boldsymbol{x} = (u, v, 1)^T, \quad \boldsymbol{x}' = (u', v', 1)$$

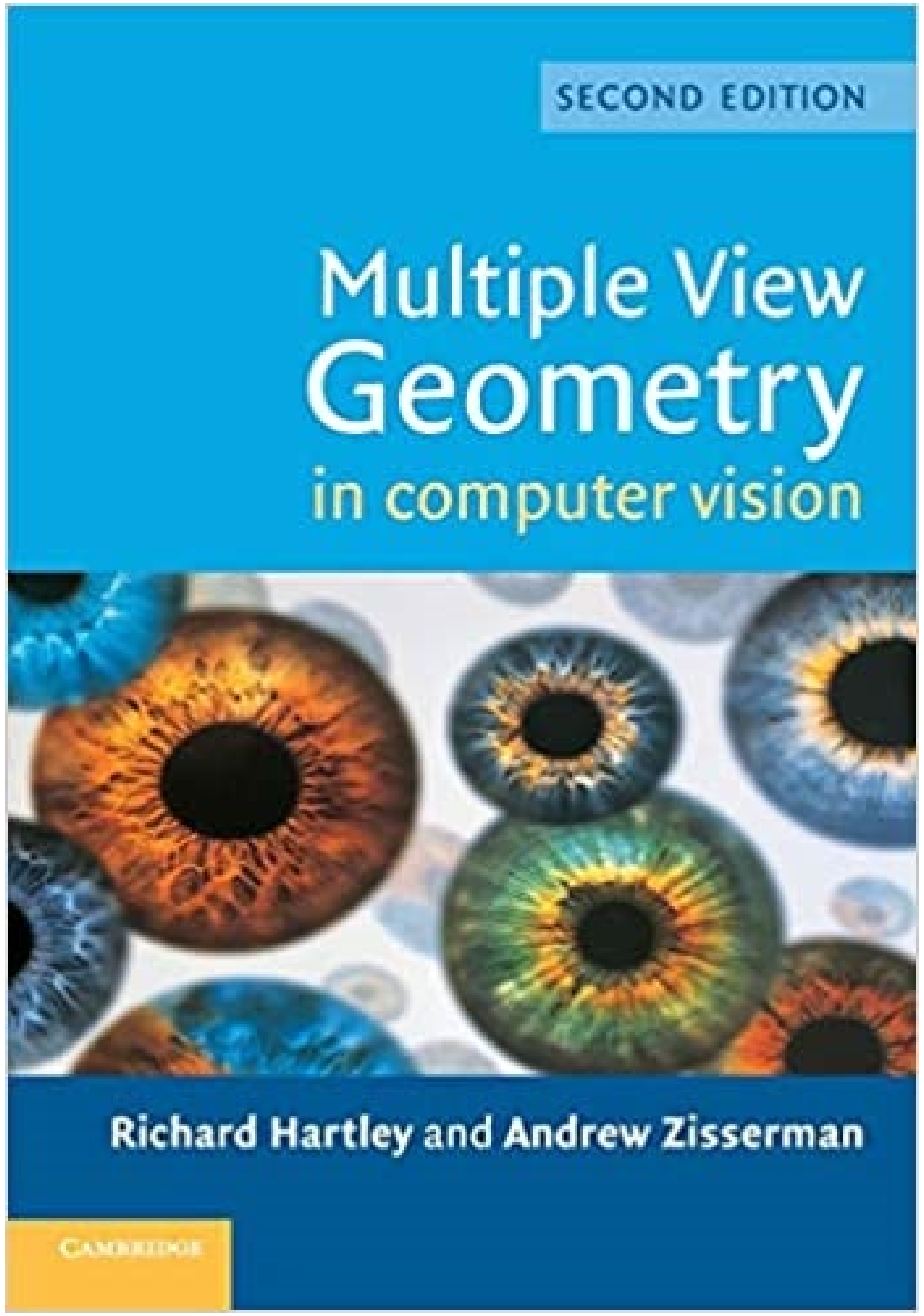
$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

Solve homogeneous linear system using eight or more matches

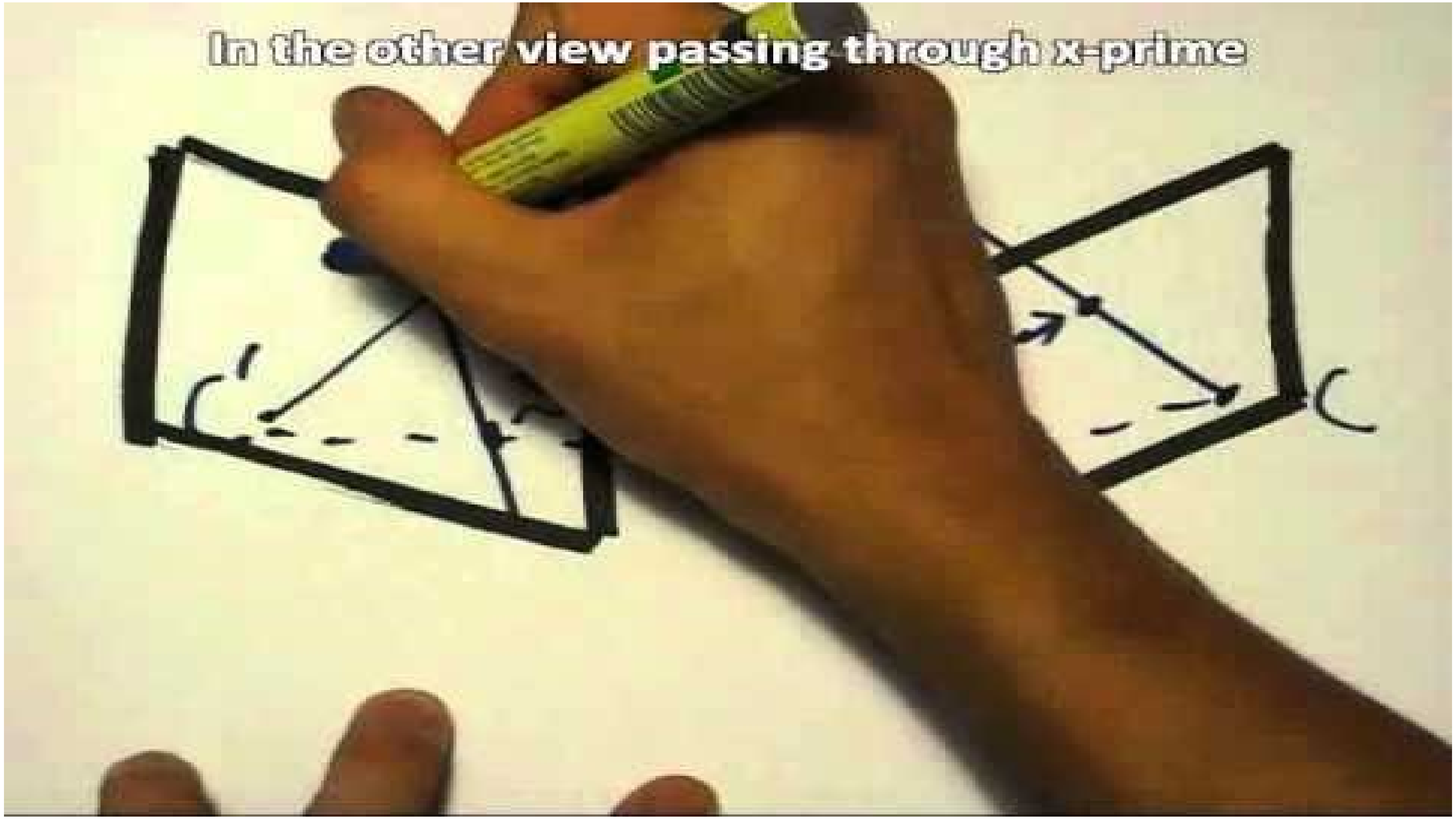
Enforce rank-2 constraint (take SVD of ***F*** and throw out the smallest singular value)



The Bible by Hartley & Zisserman



The Fundamental Matrix Song



http://danielwedge.com/fmatrix/
https://www.youtube.com/watch?time_continue=8&v=DgGV3l82NTk&feature=emb_title