2D Transformations and Warping

Discussion #4

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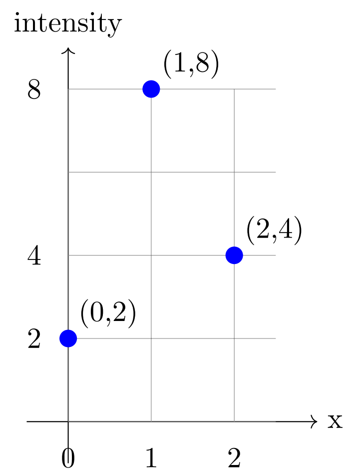
Topics

This section covers 2D transformations, interpolation, and image warping.

1 Interpolation

Problem 1.1: ***1D Linear Interpolation.***

Consider a 1D signal with 3 sample points at locations 0, 1, and 2 with values 2, 8, and 4 respectively:



The signal is sampled at the following x locations using linear interpolation. What should the interpolated values be?

Location 0.0: ________

Location 0.5: ________

Location 1.75: ________

Linear Interpolation Formula: For two points (x_1, y_1) and (x_2, y_2), the interpolated value at position x is:

$$y = y_1 + \frac{x - x_1}{x_2 - x_1} \cdot (y_2 - y_1)$$

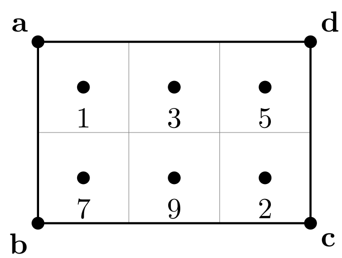
Location 0.0: This is exactly at sample point (0,2), so the value is **2**
Location 0.5: Interpolate between (0,2) and (1,8):

$$y = 2 + \frac{0.5 - 0}{1 - 0} \cdot (8 - 2) = 2 + 0.5 \cdot 6 = \mathbf{5}$$

Location 1.75: Interpolate between (1,8) and (2,4):

$$y = 8 + \frac{1.75 - 1}{2 - 1} \cdot (4 - 8) = 8 + 0.75 \cdot (-4) = 8 - 3 = \mathbf{5}$$

Problem 1.2: ***2D Bilinear Interpolation.*** Consider a 2×3 grayscale image (2 rows, 3 columns) with the following pixel values (The numbers (1, 3, 5, 7, 9, 2) represent the pixel values at their respective centers):



Part (a): What are the (x, y) coordinates for points a, b, c, and d?

Based on the 2×3 image coordinate system:

- Point **a** (top-left): $(0, 0)$
- Point **b** (bottom-left): $(0, 2)$
- Point **c** (bottom-right): $(3, 2)$
- Point **d** (top-right): $(3, 0)$

Part (b): The image is sampled at the following locations. What should the bilinearly interpolated values be?

Location (0.5, 0.5): __________

Location (0.5, 0.0): __________

Location (1.5, 1.0): __________

Solutions:

- **Location (0.5, 0.5):** This is exactly at pixel center (0.5,0.5)→1, so the value is **1**
- **Location (0.5, 0.0):** This is outside the image boundary (top edge), so we clamp to the nearest pixel value: **1**
- **Location (1.5, 1.0):** This point is exactly at x=1.5, so we interpolate between the two closest pixels vertically: (1.5,0.5)→3, (1.5,1.5)→9. Since y=1.0 is halfway between 0.5 and 1.5: $f(1.5, 1.0) = 3 + 0.5 \times (9 - 3) = \mathbf{6}$

2 Warping

Problem 2.1: ***Reverse Mapping with Inverse Transform***

We have a 2×3 source image with the following pixel values:

1	2	3
4	5	6

We want to apply transformation $T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (scale by 2).

Part (a): Find the inverse transformation T^{-1}.

$$T^{-1} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}$$

Part (b): For output pixel (3,1), use T^{-1} to find which source pixel to sample from.

$$T^{-1} \begin{bmatrix} 3.5 \\ 1.5 \end{bmatrix} = \_\_\_\_\_\_\_\_\_\_$$

Source coordinates: ________

Pixel value: ________

Part (c): For output pixel (2,2), use T^{-1} to find which source pixel to sample from.

$$T^{-1} \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} = \_\_\_\_\_\_\_\_\_\_$$

Source coordinates: ________

Part (d): For (c), show how to use bilinear interpolation to get the pixel value.

Which 4 source pixels are involved? ____________________

What are their values? ____________________

Interpolated value: __________

Part (e): Why do we need the inverse transformation for reverse mapping?

Part (a): $T^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$

Part (b): $T^{-1} \begin{bmatrix} 3.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 1.75 \\ 0.75 \end{bmatrix}$ $\rightarrow$ Source coordinates (1.75, 0.75) $\rightarrow$ Nearest pixel center (1.5, 0.5) $\rightarrow$ Value 2

Part (c): $T^{-1} \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.25 \end{bmatrix}$ $\rightarrow$ Source coordinates (1.25, 1.25) $\rightarrow$ This is between pixels, need interpolation

Part (d): For (1.25,1.25): 4 nearest pixel centers: (0.5,0.5)=1, (1.5,0.5)=2, (0.5,1.5)=4, (1.5,1.5)=5.

Bilinear interpolation calculation:

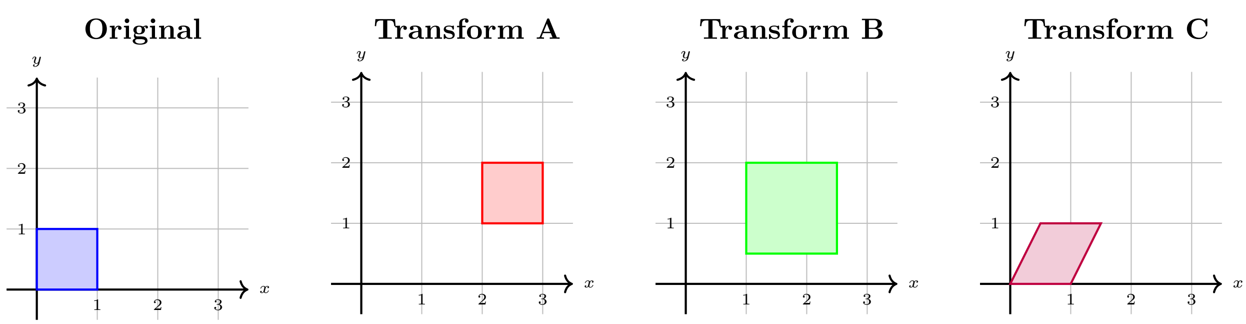
$$\begin{aligned} f(1.25, 1.25) &= f(0.5, 0.5) \cdot (1.5 - 1.25)(1.5 - 1.25) + f(1.5, 0.5) \cdot (1.25 - 0.5)(1.5 - 1.25) \quad (1) \\ &\quad + f(0.5, 1.5) \cdot (1.5 - 1.25)(1.25 - 0.5) + f(1.5, 1.5) \cdot (1.25 - 0.5)(1.25 - 0.5) \quad (2) \\ &= 1 \cdot (0.25)(0.25) + 2 \cdot (0.75)(0.25) + 4 \cdot (0.25)(0.75) + 5 \cdot (0.75)(0.75) \quad (3) \\ &= 4 \quad (4) \end{aligned}$$

Part (e): To map from output coordinates back to source coordinates for sampling.

3 Transformation Types

Problem 3.1: ***Match that transformation***

Consider the original square with vertices at (0,0), (1,0), (1,1), and (0,1). Below are three transformations applied to this square:



Match each transformation above with its corresponding transformation matrix:

Matrix I: $\begin{bmatrix} 1.5 & 0 & 1 \\ 0 & 1.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$ matches transformation ____

Matrix II: $\begin{bmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ matches transformation ____

Matrix III: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ matches transformation ____

Solution:

- Matrix I matches transformation **B** (scale by 1.5 + translate by (1, 0.5))
- Matrix II matches transformation **C** (horizontal shear by 0.5)
- Matrix III matches transformation **A** (translate by (2, 1))

Explanation:

- Transform A: Pure translation - only the translation components (2,1) are non-zero
- Transform B: Scale factor 1.5 on diagonal + translation (1, 0.5)
- Transform C: Shear matrix with 0.5 in the (1,2) position causes horizontal shearing

Problem 3.2: ***Homogeneous coordinates***

Suppose we want to translate an image by (2, 3).

Part (a): Why can't we represent translation $(x, y) \to (x + 2, y + 3)$ as a 2×2 matrix?

Translation is not linear in 2D space - it doesn't satisfy $T(ax + by) = aT(x) + bT(y)$.

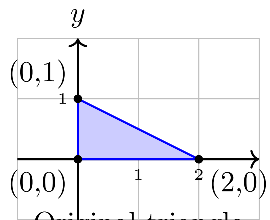
Part (b): Write the 3×3 homogeneous matrix for translating by (2, 3).

$$T = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (c): Consider the triangular object with vertices at (0,0), (2,0), and (0,1). Compare what happens when we apply transformations TR versus RT, where T is translation by (3,0) and R is 90° counterclockwise rotation.

Given: $T = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



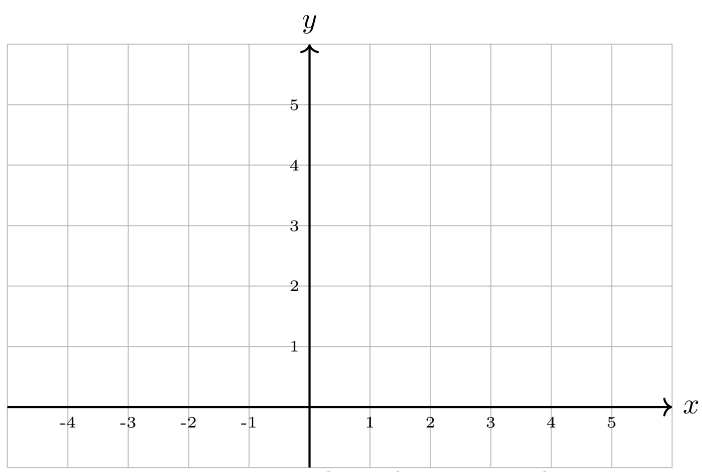
Original triangle

Calculate the composite transformations:

$TR = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$ New vertices: ______________________

$RT = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$ New vertices: ______________________

Draw both transformed triangles on the graph below:



Draw TR triangle and RT triangle

$TR = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ vertices: (3,0), (3,2), (2,0)

$RT = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ vertices: (0,3), (0,5), (-1,3)

TR: rotate first, then translate; RT: translate first, then rotate around origin

Solution graph: