Transformations and Homographies

Discussion #5

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Topics

This worksheet covers 2D transformations, focusing on the homography transformation and how to derive the linear system needed to solve for homography parameters.

1 Matching Transformations (again)

Below are matrix structures for common 2D transformations and their visual effects on a square.

Matrix Structures:

$$\text{A: } \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{B: } \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

$$\text{C: } \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{D: } \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{E: } \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Visual Diagrams:

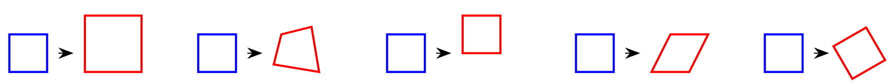


Diagram 1 Diagram 2 Diagram 3 Diagram 4 Diagram 5

Match each transformation type below to: (i) a matrix structure, (ii) visual diagram, and (iii) a degree of freedom (DoF) count.

(a) Translation:

Matrix: ________ Diagram: ________ DoF: ________

(b) Rigid:

Matrix: ________ Diagram: ________ DoF: ________

(c) Similarity:

Matrix: ________ Diagram: ________ DoF: ________

(d) Affine:

Matrix: ________ Diagram: ________ DoF: ________

(e) Projective/Homography:

Matrix: ________ Diagram: ________ DoF: ________

2 Deriving the Homography Linear System

Given a point $[x, y, 1]^\top$ in the source plane and point $[u, v, 1]^\top$ in the target plane, recall that 2D homographies follow

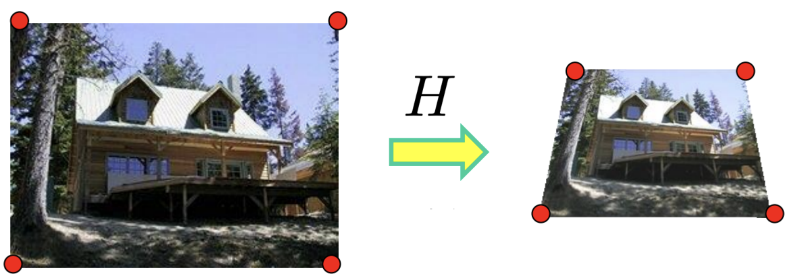
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = H \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a scalar factor and H is a 3×3 homography matrix:

$$H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & 1 \end{bmatrix}. \tag{2}$$

Homographies are defined up to scale, so we can fix $h_9 = 1$ (if we assume $h_9 \neq 0$).

Computing H. In Project 3, we will solve for H from a set of point correspondences:



This can be done by setting up an ordinary least squares (OLS) problem that can be solved with functions like `np.linalg.solve()` or `np.linalg.lstsq()`:

$$A\mathbf{h} = \mathbf{b}, \tag{3}$$

where $\mathbf{h} = [h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8]^\top$ flattens the H matrix. Equivalently,

$$\begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_8 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}. \tag{4}$$

Goal. We can solve for H by instantiating $\mathbf{a}_i^\top$ and b_i for each row i. Let's give this a try!

Problem 2.1: Single Correspondence

We'll start by setting up a system of equations. Consider a single correspondence pair: source point $[x, y, 1]^\top$ maps to target point $[u, v, 1]^\top$:

$$\begin{bmatrix} \lambda u \\ \lambda v \\ \lambda \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{5}$$

(a) We can start by algebraically eliminating the λ term. Write expressions for u and v in terms of $h_1, \ldots, h_8$, and x, y. (hint: division)

(b) Convert your two equations from Part **(a)** into linear equations in the form: $(...)h_1 + (...)h_2 + ... + (...)h_8 = (...)$. Why is this useful?

(c) What $\mathbf{a}_1^\top$, $\mathbf{a}_2^\top$, b_1, and b_2 values does this correspondence pair produce?

Problem 2.2: Multiple Correspondences

(a) Suppose we have multiple correspondence pairs: $(x_1, y_1) \to (u_1, v_1)$, $(x_2, y_2) \to (u_2, v_2)$, $(x_3, y_3) \to (u_3, v_3)$, $(x_4, y_4) \to (u_4, v_4)$. Fill in the blanks in this matrix system for 4 correspondence pairs:

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & \_\_\_ & \_\_\_ \\ 0 & 0 & 0 & x_1 & y_1 & 1 & \_\_\_ & \_\_\_ \\ x_2 & y_2 & 1 & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \end{bmatrix} = \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}$$

(b) How many point correspondences do we need at minimum to solve for the homography H? Why?

(c) Can we still solve the system if we have more than the minimum number of correspondences? What happens?

(d) How many point correspondences are needed to solve for each of the following 2D transformations?

Translation: ____________

Rigid: ____________

Similarity: ____________

Affine: ____________